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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Н. М. Бескин

ИЗОБРАЖЕНИЯ ПРОСТРАНСТВЕННЫХ ФИГУР

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IMAGES OF GEOMETRIC SOLIDS

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CHAPTER 1

Theory

1. The subject matter of the theory of images. Drawing a plane figure is not geometrically difficult because the image drawn is either an exact copy of the original or a similar figure, e.g. the drawing of a circle looks like the original circle.

Drawing geometric solids is quite a different matter. Unfortunately, there are no "spatial pencils" which can trace an object in the air. Such a pencil would "draw" a cube by tracing along its edges. Hence, we have to sketch a cube on paper with an ordinary pencil. A plane image will never be an exact copy of a solid and, therefore, a certain routine ought to be followed in drawing a solid that would create an image of the original in *the best way*. The meaning of "the best way" is discussed in the next section.

2. Requirements of an image. There are two requirements: obviousness and easy measurability. Obviousness means that an image should visually *resemble* the original. This implies the resemblance of geometric forms rather than any non-geometric property, for example colour.

Easy measurability means having the ability to determine the dimensions of an original with the minimum of effort.

The two requirements contradict each other. This is the reason why descriptive geometry, which deals with representing geometric solids on a plane, has developed various techniques that either make a compromise between obviousness and easy measurability or give priority to one of them. The choice of a technique depends on the purpose of the drawing.

Obviousness is absolutely vital in a picture while easy measurability is not important. The message conveyed by an artist can be understood, in greater or less degree, without the application of mathematics.

An engineering drawing is a different thing. Here easy measurability rather than obviousness is of major importance because the drawing will be given to a craftsman who will manufacture the object.

Some methods of drawing are far from being obvious. A layman will not be able to understand an engineering drawing because he does not have the special training, whereas a specialist will easily be able to determine all the dimensions of the original from the drawing.

3. What is the book about. Descriptive geometry embraces so many methods that even a brief account would make up a rather thick volume. Therefore, we shall discuss just one of these methods, so as to enable the reader to make stereometric drawings and solve the respective problems.

A school pupil studying stereometry sketches objects without observing any rules. He usually copies the sample drawings either given in his textbook or those drawn on the blackboard by the teacher. This book presents a geometric theory of constructing stereometric drawings. Having mastered this theory, a reader will be able to make the drawings himself rather than have to stick to the few sample ones.

The first chapter presents the theory, the second one is devoted to its applications (drawing of a cube, a cone, a cylinder, etc.), and the third one describes a method of plotting the points of an image if their coordinates are known.

4. The method of parallel projection. Proper projection methods ensure obviousness, and the central projection and parallel projection methods are the easiest.

Figure 1 shows a central projection. Let us fix point S (the **centre of projection**) and plane π (the **plane of projection**) which does not contain S. We now draw a line SA' (the **projecting line** or **projector**) through point A' in space. Point A, i.e. the point where the line pierces plane π, is the **projection** of A'. Figure 1 shows the projections of A' and B'. When all or some of the points of the original object have been projected onto the plane the drawing obtained is enlarged or reduced and an **image** is produced.

Central projection produces the most obvious images since it simulates the process of vision. Figure 2 shows that the eye does not discriminate between light rays coming from the points on the original (A', B', ...) and those coming from their respective pro-

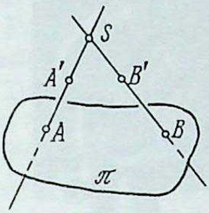

FIG. 1

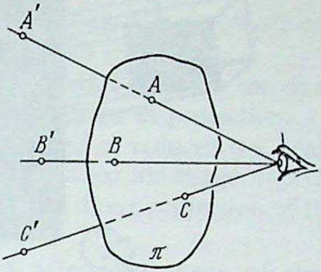

FIG. 2

jection (A, B, ...) if it visualizes the original and its projection on a plane (π).*

Artists use the method of central projections.

Parallel projection is distinct from central projection only in that the projecting lines do not pass through a common point since they are all parallel to the same direction (Fig. 3).

Images obtained by parallel projection are slightly less obvious because the technique does not simulate the process of vision so well; still they are quite obvious, the original is easily recognized because rays of vision tend to be parallel if the eye-to-original distance tends to infinity. An image produced by parallel projection resembles a small object visualized from afar.

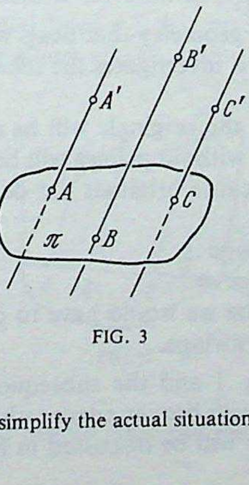

FIG. 3

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* Here we somewhat simplify the actual situation since an original is seen *by two eyes.*

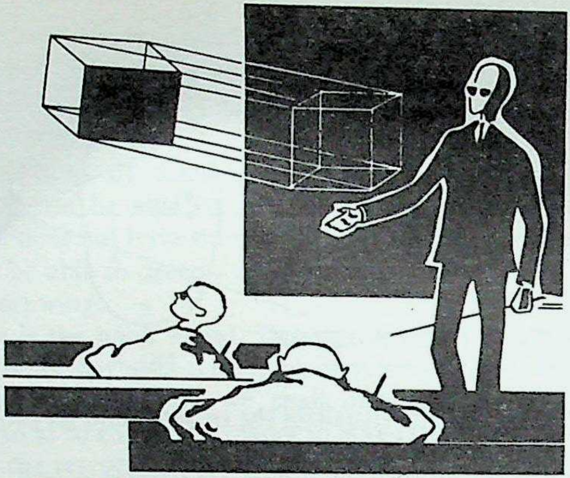

FIG. 4

Parallel projection is easier to perform than central projection, which is why parallel projection is always used when illustrative drawings are made for educational and scientific publications.

5. A comment on notation. We are dealing with images rather than with their originals when we sketch stereometric drawings. A teacher says: "This is a cube", having made its drawing. But, in fact, it is the image of a cube. Where is the cube itself? It is somewhere above our heads (Fig. 4). Projecting lines pass through the cube's vertices and pierce the blackboard at the points marked by the teacher.

The only branch of geometry that deals with originals is called descriptive geometry. It investigates the relationships between the original and its image.

All the elements of the originals will be designated by primed letters, and the letters without primes will be used for the images. This is convenient because originals are dealt with very seldom. For example:

point A' has an image A;
line m' has an image m.

If we did it otherwise we would have to prime all the letters in all the stereometric drawings.

The drawing in Fig. 1 and the subsequent ones show planes somewhat unusually (with broken edges rather than in the form of a parallelogram). This will be discussed in Sec. 19.

6. Properties of parallel projections. The procedure of parallel projection consists of two sequential steps:

1. Points of the original are carried along in the direction m **(direction of projection)** onto the plane of projections.
2. The obtained figure is enlarged or reduced (i.e. scaled up).

The result of these two steps is called the **image.** It is clear now that the point of an image is not a direct projection of the corresponding point of the original.

The second step does not affect the shape of the figure. Naturally, it may be omitted in some cases.

Points on an original contained by the same projecting line are called **concurrent points.** *Concurrent points have the same image point.*

The main properties of parallel projections are:

Property 1. *The image of a line is either a line or a point.*

Suppose that a' is not a projecting line. Mark arbitrary points A', B', and C' on a' and draw projecting lines through these points (Fig. 5). These lines belong to the same plane (that contains a' and is parallel to m). The intersection of this plane with the plane π is also a line*.

Note that a plane parallel to the direction m is called a **projecting plane.**

A projecting line consists of concurrent points.

Property 2. *Parallel lines have images that are either parallel lines or coincident lines or are distinct points.*

Suppose the parallel lines a', b', and c' are not projecting ones. The projecting planes α', β', and γ' passing through them

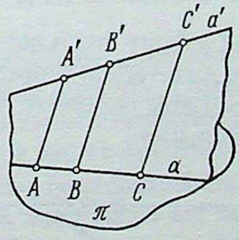

FIG. 5

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* We shall not always mention the second step (similarity transformation) because it is unimportant.

are parallel to each other (or coincide) and, henceforth, intersect plane π along parallel (or coinciding) lines. Now, if the parallel lines are projecting ones, their images are the distinct points.

Property 3. *The ratio between two parts in which a point divides the segment is the same for both the image and the original*.*

Suppose that the segment $A'C'$ contains a point B'. The lines $A'A$, $B'B$, and $C'C$ are parallel (Fig. 5), hence

$$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$

If A', B', and C' are points on a projecting line, then the three points A, B, and C coincide and the ratio $\frac{AB}{BC}$ is $\frac{0}{0}$, i.e. is indeterminate. Since $\frac{0}{0}$ may be identified with any value the equation proved above remains correct in this case as well.

Property 3 proves that the image of the midpoint of a segment is the midpoint of the segment's image.

Note 1. That B' is an internal point of the segment $A'C'$ is not essential: If B' lies on the extension of the segment $A'C'$, then we can also assume that B' divides $A'C'$ in the ratio $\frac{A'B'}{B'C'}$ and the ratio is considered negative. On this assumption *any point on the line $A'C'$ divides both the original and the image of segment $A'C'$ in the same ratio*. For example, C, D and E in Fig. 6 divide segment AB in the following ratios: $(ABC) = 3$, $(ABD) = -3$, $(ABE) = -1/3$.

Note 2. A property of a figure that is not changed by geometric transformation is called an **invariant** property with respect to this transformation. An invariable parameter of a figure is called an **invariant** of the transformation.

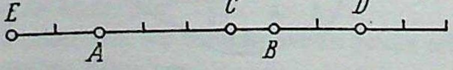

FIG. 6

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* This ratio is called the **ratio of three points** of a line and is designated (PQR), with R the point dividing the segment PQ in the ratio $(PQR) = \frac{PR}{RQ}$.

Parallel projection is a transformation that attributes any figure F' with an image figure F (projection).

The three statements proved above establish that rectilinearity (the property of a line) and parallelism (the property of two lines) are invariant properties of parallel projection, while the ratio of three points of a line is an invariant of parallel projection.

This is true not only for direct projection but also for constructing by parallel projection (see the footnote on page 11).

7. Free images. Two different problems can be posed when constructing images.

Problem 1. Given an original, say a cube with a 1 m long edge, and all the projecting parameters are known, e.g. the direction of projection m is parallel to a cube diagonal and inclined to the projection plane π at an angle $\alpha = 60°$, and the projection is scaled up with a factor $k = 0.02$. Construct the image.

Problem 2 differs from the first one in that the projecting parameters are not given. The problem is defined as follows: construct the image of a cube. In spite of the apparent simplicity of the formulation its meaning ought to be clarified.

When a stereometric image of the cube is being drawn, the position of the original with respect to the sheet of paper or the blackboard is not essential. One need only be certain that the finally drawn figure is the image of *any* cube.

We would like to explain this in another way. Suppose a teacher has drawn *anyhow* a cube on the blackboard (Fig. 4). Would it be possible to place a cube in the air so that its projection onto the blackboard parallel to some direction coincides with the figure? If it is not possible to do this, the figure that had been drawn would not be the image of any cube. Such a drawing would be *incorrect* and *not obvious*.

If there is just one original cube corresponding to the drawn figure, then the drawing is *correct*. Correctness is a necessary but not a sufficient condition of obviousness. Obviousness needs two more conditions which will be discussed in Sec. 22.

An image constructed regardless of the position of the original is called a **free image.** Free images are used for illustrations and for technical applications. We always draw *some* cube or *some* sphere. Therefore, *this book considers only free images*. We must follow certain rules to construct correct free images. The statement of these rules is the objective of this book.

8. Constructing the images of plane figures. Obtaining the images of plane figures means that the figures from one plane are cast onto the plane of the drawing. However, they might become distorted. In solid geometry, we have often to represent geometric figures containing various plane elements, e.g. faces of a polyhedron are drawn in the same plane of the drawing.

The theory of constructing the images of plane figures is based on the following two theorems.

Theorem 1. *Any given triangle can have another triangle as its image.*

Explanatory note. Say we are given the triangle $A'B'C'$ with $\measuredangle A' = 60°$, $A'B' = 3$ m and $A'C' = 2$ m. We can draw an *arbitrary* triangle ABC and state that it is the image of $A'B'C'$.

Proof. Given two triangles $A'B'C'$ (the original) and ABC (the image). Pass the plane π, which is distinct from the plane of $A'B'C'$, through the side $A'B'$ (Fig. 7). Construct the triangle $A'B'C_1$ on $A'B'$ which is similar to ABC (there are two such triangles, and only one is considered). Let $m \equiv C'C_1$ be the projecting direction, then $A'B'C_1$ is the projection of $A'B'C'$ onto π. Similarity transformation of $A'B'C'$ generates the triangle ABC.

Theorem 2. *An image of the triangle $A'B'C'$ specifies unambiguously the image of every point in the plane that contains this triangle.*

Proof. Given the original $A'B'C'$ and the corresponding image ABC (Fig. 8). Choose a point D' in the plane of $A'B'C'$ and join this point to a vertex of $A'B'C'$, e.g. A'. Designate the point where $A'D'$ intersects the opposite side $B'C'$ as E' (it is not important whether E' is inside $B'C'$ or outside it on its extension). $A'D'$ may be parallel to $B'C'$, but suppose that $A'D'$ is not parallel to $B'C'$. Then the image E of the point E' may be con-

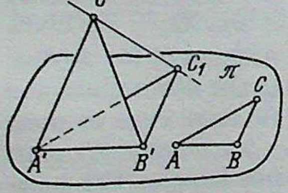


FIG. 7

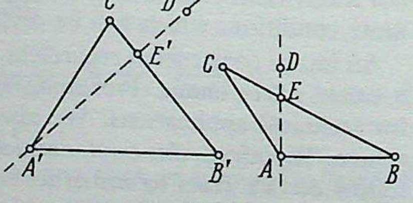


FIG. 8

structed: it divides the segment BC in the same ratio as E' divides the segment $B'C'$:

$$\frac{BE}{EC} = \frac{B'E'}{E'C'}.$$

Point D should lie on the line AE. The position of D can be determined from the following ratio:

$$\frac{AD}{DE} = \frac{A'D'}{D'E'}.$$

If $A'D' \parallel B'C'$, then $AD \parallel BC$ and

$$\frac{AD}{BC} = \frac{A'D'}{B'C'}.$$

A practical rule for constructing the images of plane figures stems from the two theorems proved above. The image of a plane figure may be drawn *arbitrarily up to a certain moment* after which a strict construction procedure must be followed because nothing may then be done in an arbitrary way.

The theorems we have proved help register this crucial moment: any three unspecified points (i.e. not contained in the same line) may be arbitrarily chosen and three random unspecified points ascribed as their images. This is the limit of arbitrariness since the images of all the other points should be constructed. In other words, it is possible to give the image of a triangle in an arbitrary way; it is impossible to do this for a quadrilateral.

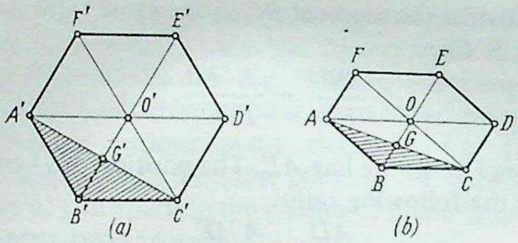
9. Some examples of representing polygons. Example 1. *Construct the image of a square.* Note that the image of a parallelogram is a parallelogram. On the other hand, any given parallelogram (e.g. a square) can have any parallelogram as its image. In fact, the triangle $A'B'C'$ may be singled out of the parallelogram $A'B'C'D'$ and assigned an arbitrary triangle ABC as its image which is then completed to a parallelogram.

Example 2. *Construct the image of a regular hexagon.* The construction of the image of a plane figure can be divided into three stages.

Stage 1. Imagine or draw the original as it is (i.e. undistorted).

Stage 2. Single out a triangle inside the original and assign an arbitrary triangle to it as its image.

Stage 3. Construct the remaining elements of the figure sequentially using their links with those already drawn. But (Attention!



(a) (b)

FIG. 9

This is very important.) you may only use those links that are invariant with respect to parallel projection. For instance, the images of two perpendicular lines may not be perpendicular while the images of parallel lines are always parallel.

Let us solve the problem defined above. Figure 9 displays a regular hexagon $A'B'C'D'E'F'$ as it is (stage 1). Draw an arbitrary image of the triangle $A'B'C'$ (stage 2, Fig. 9*b*). G' is the midpoint of the segment $A'C'$, hence G is the midpoint of AC since this property is invariant. Now draw the line BG and mark the points O and E on it using the invariant ratios $B'O' = 2B'G'$ and $B'E' = 4B'G'$. Then draw $C'D' \parallel B'G'$ and $A'F' \parallel B'G'$. D may be found by drawing either $CD = BO$ or $ED \parallel AB$ or AO. F may be constructed in the same way.

10. The image of a circle. Example 3. *Construct the image of a circle.*

Explanatory note 1. The image of a circle is an ellipse*.

Explanatory note 2. The curve is plotted by points. There are instruments for drawing curves such as compasses for circles. Since there is not an instrument for drawing ellipses, the construction of an ellipse boils down to plotting the required number of points.

The use of conjugate diameters for constructing an ellipse is discussed in Sec. 32. Here, we shall consider the problem of *drawing the image of a circle, given its three arbitrary points*, a task more closely related to the topic being discussed. This problem may be formulated another way, viz. *draw the image of a circle circumscribed about a triangle.*

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* The reader can find some more information about ellipses in Appendix 2 (p. 67).

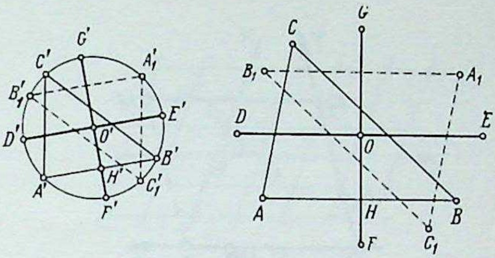

FIG. 10

Figure 10 shows the original (on the left): a circle with a triangle $A'B'C'$ inscribed in it. Let us cast the image of $A'B'C'$ as the arbitrary triangle ABC (on the right of Fig. 10) and construct the point O, the image of the centre of the circle (the construction is not shown here).

If a point, symmetric with respect to the centre O', is constructed for each vertex of the original triangle, then a new triangle $A_1'B_1'C_1'$ is formed which is symmetric to $A'B'C'$. Its vertices are also contained in the circle. The image of $A_1'B_1'C_1'$ can be easily constructed: just find the points A_1, B_1, and C_1 symmetric to A, B, and C with respect to O and we thereby obtain three more points of the image of the circle.

Lines symmetric with respect to the centre are parallel to each other. Hence, $A'B' \parallel A_1'B_1'$. Draw two diameters $D'E' \parallel A'B'$ and $F'G' \perp A'B'$ so that they pass through O'. The images of these lines can easily be constructed: $DE \parallel AB$ and FG passes through the midpoints of AB and A_1B_1. Their end points D, E, F, and G can be determined from the following relations

$$\frac{OD}{AB} = \frac{O'D'}{A'B'}, \qquad \frac{OF}{OH} = \frac{O'F'}{O'H'}$$

(H' is the point where $F'G'$ intersects $A'B'$, and H is the midpoint of AB).

Thus, we have constructed two conjugate diameters, DE and FG, of the ellipse which are the images of the two perpendicular diameters $D'E'$ and $F'G'$ of the circle. An ellipse can be constructed from its conjugate diameters (see Sec. 32).

11. Another viewpoint of constructing the images of plane figures. Every point can be plotted, given a fixed coordinate system.

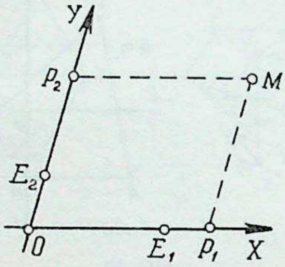

FIG. 11

An affine coordinate system (in the plane) consists of the following elements (Fig. 11):

(1) Two lines intersecting each other at an origin O.

(2) A positive direction is defined for each line and shown by an arrow.

A line with one specified direction (of the two available) is called **oriented line** or **axis.** Therefore, (1) and (2) may be replaced by a single element, i.e. two intersecting axes.

(3) The order of axes is defined. This means that one of them is the first and the other one is the second. The order is denoted either by letters X and Y or by numbers *1* and *2* or by colours, e.g. black and red, etc., it is hardly possible to list all the adequate notation. It is only important that the axes can be distinguished from each other.

(4) A **unit scale** is defined on each axis. This is done by marking **unit points** E_1 **and** E_2.

Given an affine coordinate system, each point M of the plane can be defined by **affine coordinates.** Draw the lines MP_2 and MP_1 through M parallel to the X- and Y-axes, respectively. The affine coordinates of M are the numbers:

$$x = \frac{OP_1}{OE_1}, \qquad y = \frac{OP_2}{OE_2},$$

the sign being assigned in accordance with the well-known rule. Note that coordinates are abstract (dimensionless) numbers rather than geometric segments.

If the X- and Y-axes are perpendicular to each other and their unit scales are identical, i.e.

$$\angle XOY = 90^\circ, \qquad OE_1 = OE_2,$$

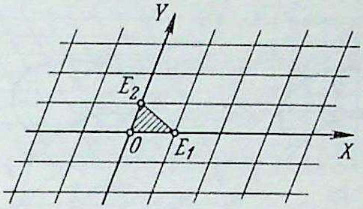

FIG. 12

then the coordinate system is called a **cartesian system** and the x and y coordinates, **cartesian coordinates.**

A coordinate system has what is called a coordinate grid (Fig. 12). The plane is ruled into identical parallelograms parallel to the X- and Y-axes. The sides of the parallelograms are the unit scales of the axes.

The coordinates of any point M are evident from its position within the grid. We will skirt the technical difficulty encountered if M is not a **nodal point** (i.e. a vertex of a parallelogram) and its coordinates are estimated visually. To facilitate this procedure the grid lines should be spaced closer together (e.g. each line corresponds to one tenth of the scale unit).

A coordinate grid has as its image a similar coordinate grid consisting of identical parallelograms. Only the parallelograms (their side lengths and angles) vary.

The image of a coordinate grid defines the image of any point in the plane. Each point M' has an image M, both points being located within their respective coordinate grids. In other words, M (*the image*) *is defined by the same coordinates as* M' (*the original*) *but the coordinates of* M' *are related to the natural coordinate system* (this is the common term for the original system), *while the coordinates of* M *are related to the image of the natural system.*

Figure 13*a* shows a figure F' superimposed on a cartesian coordinate grid. Figure 13*b* represents an arbitrary image of the coordinate system (i.e. coordinate grid) and the image F.

In the above sections it was stated that only the image of a triangle may be defined *arbitrarily*, everything else is *routine*. This section is entitled 'Another viewpoint ...'. It states that the image of a coordinate system may be arbitrarily chosen.

We now invite the reader to have a close look at Fig. 11 in order to discover a very important fact that *a triangle with specified ver-*

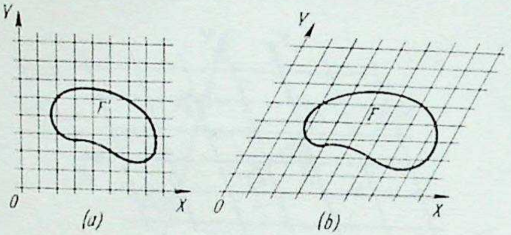

FIG. 13

tices (i.e. denoted either by letters or numbers or otherwise) *defines an affine coordinate system*. The triangle OE_1E_2 in Fig. 12 evidently defines a coordinate grid as this grid may be easily constructed from the given points O, E_1, and E_2.

12. Pohlke-Schwartz theorem. We know that a triangle plays a special role when drawing plane figures. We also understand the nature of this role: a triangle (with specified vertices) is itself a coordinate system. It is the nucleus of a coordinate grid. Had everything been erased from Fig. 12 except for the three points O, E_1, and E_2, these points would have been enough to restore the whole drawing again.

Those who have learnt this may realize that a tetrahedron has to play a similar role for representing three-dimensional objects.

Why do we think so? The reason why is that a tetrahedron with specified vertices is a spatial affine coordinate system (we are not explaining what spatial affine coordinate system means because it must be clear to the reader who is now familiar with the plane affine system). It is the nucleus of a spatial affine coordinate grid. If points O, E_1, E_2, and E_3 (Fig. 14) are all that remained from the whole coordinate grid, it would be enough to restore the grid.

These considerations prompt the following statement (Pohlke-Schwartz* theorem).

Theorem 3. *Any tetrahedron can be represented as any arbitrary complete quadrilateral.*

Note 1. A quadrilateral with diagonals is called a complete quadrilateral. A more accurate definition reads that a complete quadrilateral is a plane figure consisting of four points in some

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* Pohlke proved the theorem in 1853 for an isosceles right tetrahedron (this has equally long edges and a right angle between the faces at the apex). Schwartz proved the theorem for an arbitrary tetrahedron in 1864.

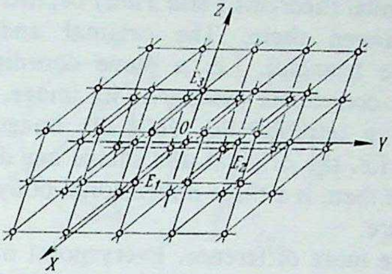

FIG. 14

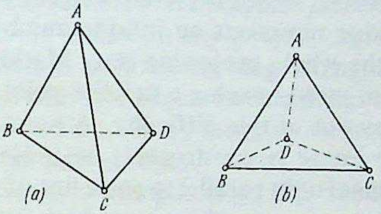

FIG. 15

general position (i.e. no three lie on one straight line) and six segments joining these points in pairs.

The quadrilateral in this theorem is not necessarily convex. Figure 15*a* represents tetrahedron *A′B′C′D′* as a concave quadrilateral, *ABCD* with diagonals *AC* and *BD*. Figure 15*b* depicts the same tetrahedron as a non-convex quadrilateral *ABCD* (the reader is invited to draw it separately without the diagonals) with diagonals *AC* and *BD*.

Note 2. Do not think that the broken lines are diagonals. These lines are used for showing invisible lines (it is assumed that the sides of a polyhedron are opaque). Whatever the positions of points *A*, *B*, *C*, and *D*, the segments *AB*, *BC*, *CD*, and *DA* in quadrilateral *ABCD* are edges and *AC* and *BD* are diagonals.

Note 3. Remember that Theorem 3 is just a guess that will be proved later, therefore we will call it a theorem until then.

And now, having formulated the theorem and having supplied provisos against any possible misunderstanding, we may work on the proof. Nevertheless, we shall postpone the proof for a while because pondering over the logic of the theorem might help to clarify the whole idea even more than obtaining the proof itself.

However similar theorems 1 and 3 may be, there is an important difference between them. The original and the image in Theorem 1 are triangles, i.e. a plane coordinate system has another plane coordinate system as its image. The original in Theorem 3 is a tetrahedron, while its image is a complete quadrilateral, i.e. the original and image are different types of figures, one of them is a three-dimensional body, while the other is a plane figure.

There is one more difference. Every point of the image of a plane figure corresponds to a *single* point of the original. This means that one point of the original may be shown just by marking it on the drawing. When depicting objects, each point of the plane of a drawing represents an infinite number of concurrent points, i.e. of the whole projecting line. Marking a point on a drawing does not mean imaging a definite point of the original.

There is a way out of this difficulty. A point M' should first (i.e. before the image is constructed) be projected from some point of the original or in parallel to some line of the original onto a plane of the original (this procedure is called **internal projecting**). Designate the point thus obtained M_0'. Next, M and M_0 are constructed as the images of M' and M_0'. These two points define the position of M' in space. Point M_0 is called the **secondary projection** (though secondary image is a better term) of M'.

For example, Fig. 16 has been constructed as follows: first, M' was projected in parallel to the Z'-axis onto the $X'Y'$-plane, then the image of the whole of this was cast onto the plane. The point M in Fig. 16 is the image of M', and M_0 is its secondary image.

The point E in Fig. 19*b* (cf. p. 26) is the secondary projection of M'.

Let us consider Theorem 3 again. Its proof is based on the following lemma.

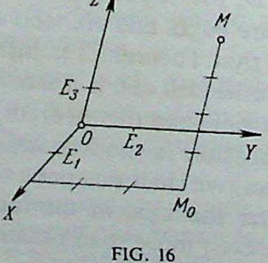

FIG. 16

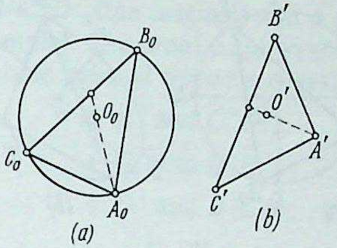

FIG. 17

Lemma. *The cross section of a triangular prism on a plane may be a triangle which is similar to any given triangle.*

A triangular prism is an infinitely long 'triangular tube' rather than a polyhedron with five edges. Such a prism may be defined by its normal cross section, i.e. by the cross section cut by a plane perpendicular to the lateral edges.

Now, we are given two triangles (Fig. 17). The triangle $A_0B_0C_0$ is the normal cross section of the prism, while the triangle $A'B'C'$ is the sample. We need to cut the prism by a plane in order to get a triangle similar to $A'B'C'$.

Now we are going to do some analysis. Suppose α is a plane perpendicular to the prism's edges, and β is the plane sought. The triangles $A_0B_0C_0$ and $A''B''C''$ are the cross sections of the prism by α and β, respectively, $A''B''C''$ being similar to $A'B'C'$.

Consider the parallel projection of α on β with the projection direction parallel to the prism's edges. $A_0B_0C_0$ is projected onto $A''B''C''$. The circle circumscribing $A_0B_0C_0$ (Fig. 17) is projected onto the ellipse circumscribing $A''B''C''$. The projecting lines form a *cylinder circumscribing the prism.* Scale up $A''B''C''$, together with the circumscribing ellipse so that it is converted into $A'B'C'$. Then the ellipse will be transformed into the ellipse circumscribing $A'B'C'$.

Triangle $A'B'C'$, together with the ellipse, is the image of $A_0B_0C_0$ and the circle circumscribing it because the former was obtained from the latter by parallel projection and by a subsequent similarity transformation. That being the case, the ellipse circumscribing $A'B'C'$ is defined and may easily be constructed by plotting points (this procedure is described in Sec. 10, see Fig. 10). Figure 17*b* shows the centre of the ellipse.

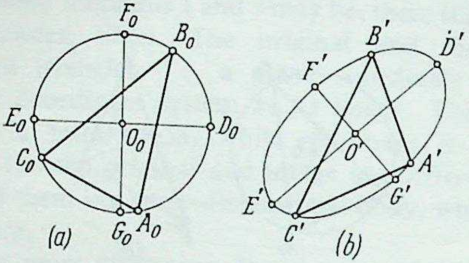

FIG. 18

We draw the axes of the ellipse (its major axis $D'E' = 2a'$ and its minor axis $F'G' = 2b$ (Fig. 18)) and construct the two mutually perpendicular diameters D_0E_0 and F_0G_0 that correspond to the axes (i.e. the diameters of the circle whose images are the axes of the ellipse). We are going to show how a figure (the ellipse with its inscribed triangle) which is similar to that constructed above can be 'set' upon a cylinder.

If a circular cylinder is intersected by a plane the resulting cross section is an ellipse with

(1) a minor axis equal to the diameter of normal cross section, and

(2) the ratio of its semi-axes equal to the cosine of the angle between the intersecting and normal planes (see Sec. 31).

Now, bearing this in mind, let us

(1) draw a *line* parallel to F_0G_0 on a plane α, and

(2) pass a plane β through this line at the angle $\varphi = \arccos\frac{b'}{a'}$ to the plane α. Note that there are two such planes because the angle φ may be constructed on both sides of α.

Plane β, thus constructed, forms an elliptical cross section by intersecting the cylinder. End points F'' and G'' of its minor axis lie 'above' F_0 and G_0 (on their respective generatrices). Points A'', B'', and C'' are contained along the same generatrices as the points A_0, B_0, and C_0. In fact, triangle $A''B''C''$ is situated with respect to the 'cross' ($D''E''$, $F''G''$) in the same way as $A_0B_0C_0$ is situated with respect to the 'cross' (D_0E_0, F_0G_0). More precisely, point A'' has (and this is also true for B'' and C'') the same affine coordinates in the ($O''D''$, $O''F''$)-coordinate system as point A_0 has in the (O_0D_0, O_0F_0)-coordinate system.

Now we are ready to prove Theorem 3. We are given a tetrahedron $A'B'C'D'$ (the original) and a plane quadrilateral $A^0B^0C^0D^0$ (the sample). The proof employs the following idea. A tetrahedron has three pairs of opposite edges:

$$A'B' \text{ and } C'D',$$
$$A'C' \text{ and } B'D',$$
$$A'D' \text{ and } B'C'.$$

The opposite edges of the tetrahedron are skew, while the respective lines that form the plane quadrilateral intersect. So, any plane figure has three points (called the **diagonal points** of a complete quadrilateral) which a three-dimensional body does not. Denote

the intersection of A^0B^0 and C^0D^0 as P^0,
the intersection of A^0C^0 and B^0D^0 as Q^0,
the intersection of A^0D^0 and B^0C^0 as R^0.

One, perhaps even two but not three, of the points may be missing if the appropriate lines are parallel.

Hence the *diagonal points define the direction of the projection*. It occurs as follows.

Determine the point P_1' that divides $A'B'$ in the ratio equal to that into which the point P^0 divides A^0B^0 and find the point P_2' on $C'D'$ in the same way, thus:

$$\frac{A'P_1'}{P_1'B'} = \frac{A^0P^0}{P^0B^0}, \quad \frac{C'P_2'}{P_2'D'} = \frac{C^0P^0}{P^0D^0}.$$

So *one* point P^0 of the sample determines *two different* points P_1' and P_2' of the original. These are *concurrent points*, i.e. they must be represented by one point. This means that $P_1'P_2'$ is a *projecting line*.

Draw lines parallel to $P_1'P_2'$ through the vertices of the tetrahedron $A'B'C'D'$ to obtain an infinite tetragonal prism. A cross section of this prism by *any* plane (with the exception of the projecting one) is a quadrilateral $ABCD$ for which

$$\frac{AP}{PB} = \frac{A^0P^0}{P^0B^0},$$
$$\frac{CP}{PD} = \frac{C^0P^0}{P^0D^0}.$$

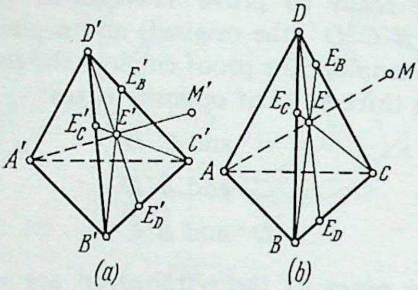

FIG. 19

Now we do not take any plane that intersects the triangular prism $A'B'C'$, we only take the one that cuts the triangular prism to yield a triangle ABC which is similar to $A^0B^0C^0$. This plane intersects the quadrilateral prism across the quadrilateral $ABCD$ which is similar to the quadrilateral $A^0B^0C^0D^0$ (the sample):

triangle ABC is similar to $A^0B^0C^0$,

$$\frac{AP}{PB} = \frac{A^0P^0}{P^0B^0},$$

$$\frac{CP}{PD} = \frac{C^0P^0}{P^0D^0}.$$

These equations show that quadrilateral $ABCD$ is similar to quadrilateral $A^0B^0C^0D^0$ and this proves Theorem 3.

13. Representing geometric solids. Theorem 4. *The image of the tetrahedron $A'B'C'D'$ defines each point in space.*

Proof. Given that $A'B'C'D'$ is an original and $ABCD$ is its image on a plane (Fig. 19), and given the point M' in space, let us construct its image. Join M' to some vertex of the tetrahedron, e.g. A', and mark the point E' where the line $A'M'$ meets the opposite face $B'C'D'$ (E' is not necessarily inside $B'C'D'$). The reader is invited to analyse the case when $A'M'$ is parallel to the plane of $B'C'D'$ by himself. By drawing $B'E'$, $C'E'$, and $D'E'$ we obtain the points E'_B, E'_C, and E'_D, respectively, on the sides of $\triangle B'C'D'$.

Now to get down to the image; the images of E'_B, E'_C, and E'_D may be constructed since they divide CD, DB and BC in the ratios equal to those of the original. By the way, it is enough to construct only two of the three points, say E_B and E_C. Point E is the intersection of BE_B and CE_C. Next draw AE and construct the point M on it so that it satisfies the following equation:

$$\frac{AM}{ME} = \frac{A'M'}{M'E'}.$$

The theorem we have proved may also be interpreted in the following way: the image of a tetrahedron may be completed to become the image of a spatial coordinate system. Figure 16 represents a spatial coördinate system that is defined by the point O (the origin) and by the points E_1, E_2, and E_3 (the images of the unit points of coordinate axes). With this image one can construct the image of any point defined by its coordinates. For instance, Fig. 16 illustrates how the image of point $M'(2, 3, 4)$ is constructed.

Example 1. *Draw a cube.*

Three of a cube's edges stemming from the same point define a tetrahedron. The Pohlke-Schwartz theorem states that the tetrahedron may have any arbitrary quadrilateral as its image. The remaining portion of the image may be completed since the cube's edges which are parallel must retain their parallelism in the image.

So, we may construct A_1, B_1, D_1, and A_2 arbitrarily when representing the cube (Fig. 20). This is what the Pohlke-Schwartz theorem actually states. Most people think that a drawing like the one shown in Fig. 20 is not always the image of a cube as the angles of the component parallelograms and the ratios of the segments must be specially chosen. Confess, reader! What was your opinion? Have you ever drawn a cube with perfect freedom or did you follow a compulsory procedure?

Another form of the Pohlke-Schwartz theorem permits us to represent a cartesian coordinate system arbitrariiy. For example, we can choose angles $\measuredangle XOY$ and $\measuredangle YOZ$ in Fig. 16 to be ar-

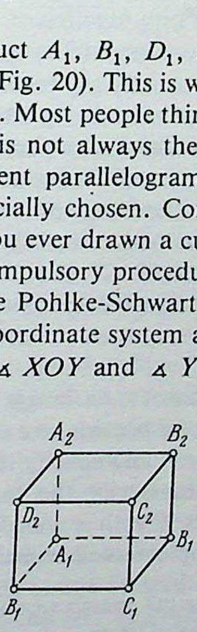

FIG. 20

bitrary and then state that axes X', Y', and Z' are perpendicular in pairs. We may also take segments OE_1, OE_2, and OE_3 of any length and state that their originals are equal and even have a specified length, e.g. $O'E_1' = O'E_2 = O'E_3 = 1$ m. This means that a lecturer of analytical geometry in space may freely draw any suitable orthogonal cartesian coordinate system.

Example 2. *Draw a regular quadrilateral pyramid.*

The base of a pyramid (a square) can be represented by any parallelogram. Besides, according to the Pohlke-Schwartz theorem, one edge can be represented arbitrarily, i.e. the vertex of the pyramid may be chosen arbitrarily.

To draw the altitude, the apex should be joined to the point where the diagonals intersect.

14. Reversibility of an image. We have been discussing the conditions for an original to define its image. But in practice the converse problem is more important, since the sole significance of the image is the information it communicates about the original. To manufacture a component, the worker should have a drawing that fully defines the component. When we look at a painting, we want to recognize the original.*

An image is called reversible if the original may be restored from it (mathematicians usually say **reconstructed**).

Let us first solve the reversibility problem for the image of a plane figure. Given the image F of a plane figure (Fig. 21*a*), take any common points A, B, and C contained in F. We know that the triangle ABC may be the image of *any* triangle. Suppose that Fig. 21*a* is *supplemented* with the following condition: in the original $A'B' = 12.4$ mm, $B'C' = 6.2$ mm, and $\measuredangle A'B'C' = 90^\circ$. With this information we may restore the triangle $A'B'C'$ precisely (Fig. 21*b*). We may then construct every point of figure F', i.e. the whole F'.

As was explained in Sec. 11, a triangle plays the role of a coordinate system. Instead of a triangle, we may superimpose a coordinate grid on the image F and specify the reconstruction of this grid. Usually, but not necessarily, the image of a cartesian grid is used. Figure 22*a* depicts F with a coordinate grid superimposed on it. Figure 22*a* is supplemented with the information that the

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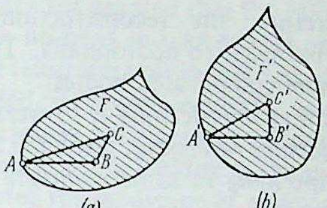
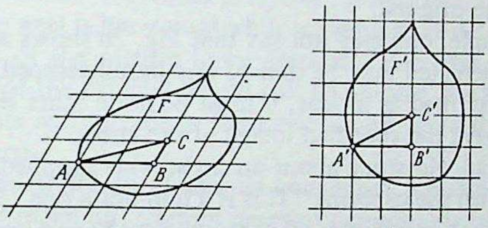
* We think that those who like abstract art will not read this book since they are not interested in any theory of image.



(a) (b)

FIG. 21



(a) (b)

FIG. 22

parallelograms of this coordinate grid represent the squares with 6.2 mm long sides. Figure 22*b* shows a natural coordinate grid with figure *F*´ on it.

We should stress once more that the image *ABC* of a triangle tells us nothing about its original (except for the fact that it is a triangle rather than some other figure). Therefore, the reconstruction of the original must be specified additionally. If the image *F* of a plane figure (more complicated than a triangle) is given, then:

(1) *this image is irreversible*, *i.e. its precise original is indefinite*;

(2) *this image becomes reversible*, *given the reconstruction of its component triangle.*

More precisely, the image of a plane figure is irreversible in any case but the following system is reversible:

(a) an image and (b) a supplementary condition that specifies the reconstruction of some triangle which is a component of the image. Given (a) and (b), the original may be precisely determined.

The images of geometric solids are similar in this respect. The original of a tetrahedron is not defined by its image. In fact, according to the Pohlke-Schwartz theorem, this image defines *any* tetrahedron. Therefore, the reconstruction of the original tetrahedron must be specified additionally. The following statement is true for more complicated bodies:

(1) *the image of a geometric solid is irreversible*;

(2) *the image becomes reversible*, *given the reconstruction of a tetrahedron corresponding to some complete quadrilateral contained in the image.*

15. Specified images. An image is **specified** if it is supplemented with certain conditions. Without these conditions it is impossible to define the original.

For example, one may not say that Fig. 20 shows a cube since the image depicted may be that of any parallelepiped. But if the image is described as a cube, it must be none other than a cube. Thus, we have the specified image of the cube.

There is an old story about an artist who supplied one of his paintings with the caption: "This is a lion not a dog"*. The joke, of course, is on the painter whose work is so bad he must supply it with an explanatory note. But in one sense the artist is right: if the caption says the picture is of a lion, then the original must have been a lion. The painter portrayed the image of the lion in a special way.

Almost all images, paintings included, are specified. Here are some examples.

In geometry, supplementary conditions are formulated explicitly, e.g., the dimensions of the relevant tetrahedron are given. In technical drawings, the sizes and the values of angles are indicated. Architects often indicate the scale of projected buildings by sketching small human figures and cars on the blue prints.

In art, supplementary conditions are always implied rather than worded (this is the reason the artist's caption 'This is a lion not a dog' sounds funny). This means that a picture always represents objects familiar to the viewer. For example, if a picture depicts rails, the viewer takes for granted that the rails are parallel* and equidistant (there is no need to take special note of the distance between them). If a telegraph pole is depicted, its perpendicularity to the rails is implied, its approximate size assumed, and so on. The viewer unconsciously imagines these conditions and restores the original in his mind while looking at the picture.

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* See the book by E. A. Vartan'yan, *From the life of words*, 2nd ed., Detgiz Publishers, Moscow, 1963 (in Russian). The same motif can be found in the folklore of other countries. For example, Don Quixote of Lamancha told the following story about a painter: "Once he painted the picture of a cock. But it was so poorly done that the cock did not look like a cock at all. So the painter had to write on it: 'This is a cock', in large letters".

If a picture depicted the landscape of an alien planet without a single familiar object, it would be impossible to get a precise idea of the original. Even in such a case, however, certain conditions would be assumed. We would perceive, for example, that the surface of the planet is horizontal while imagining the details parallel to the sides of the picture to be vertical. Of course, the actual dimensions of the original would still be unclear. Perhaps future cosmic artists will include at least one object from the Earth in their landscapes.

As mentioned above, almost all images are specified. 'Almost all' does not mean all. Certain necessary conditions must still be supplied if we want to get a precise idea of the original. But it may be, when one studies solid geometry for example, that the metric properties of an original (sides and angles) are not important.

Look at Fig. 20 again. If there are no supplementary conditions, then we may say that this is an image of *some* parallelepiped. Figure 20, as given without any supplementary conditions, is sufficient for proving theorems or solving problems dealing with arbitrary parallelepipeds.

Figure 15 without supplementary conditions may be used as an illustration for dealing with arbitrary tetrahedrons. If one wants to represent a regular tetrahedron, then Fig. 15 must be supplemented with a 'this is a lion, not a dog' condition that would read 'this is a regular tetrahedron' or '$A'B' = A'C' = A'D' = B'C' = B'D' = C'D'$'**. If it is necessary to indicate size, then one may add '$A'B' = a$' or '$A'B' = 1$ cm'.

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* Recall that a painted image is constructed in compliance with rules that differ from those expounded in this book. In a painting central rather than parallel projections are used. Therefore, the rails in the picture are not parallel.

** Do not use primes when studying solid geometry. Write $AB = AC = \ldots$ as if the image is the tetrahedron itself. We have to be pedantic in this book that is devoted to image-original relations.

CHAPTER 2

Practical Exercises

16. Cross sections of polyhedrons. Here we shall deal with the images encountered in the study of solid geometry. We shall primarily consider questions that involve metrics (i.e. the measure of segments and angles). We shall show how to construct images and analyse the most frequently made mistakes.

Example 1. *Represent a cross section of a prism by a plane.*

Figure 23*a* displays a cross section of a triangular prism. It has been 'constructed' very easily: the arbitrary points A_3, B_3, and C_3 have been chosen on the edges of the prism and joined to each other with line segments. The same 'method' has been applied to the construction of a cross section of a quadrilateral prism in Fig. 23*b*. Figure 23*a* is correct while Fig. 23*b* is not. We shall now analyse a common mistake using Fig. 23*b*.

A plane is defined by three points. Therefore, we may choose points on the edges of a triangular prism arbitrarily when constructing its plane cross section. This may not be done when dealing with quadrilateral prisms. By choosing the points A_3, B_3, and C_3 arbitrarily, we define the secant plane; the point of its intersection with the fourth edge of the prism could not be arbitrary.

But might not Fig. 23*b* be correct by chance? Is it not possible, in placing the chosen point D_3 arbitrarily, to place the point exactly where it should be? We shall show how this may be checked.

The plane that passes through the edges $A_1'A_2'$ and $C_1'C_2'$ intersects the plane cross section $A_3'B_3'C_3'D_3'$ along the diagonal $A_3'C_3'$, while the plane that passes through the edges $B_1'B_2'$ and $D_1'D_2'$ intersects $A_3'B_3'C_3'D_3'$ along the diagonal $B_3'D_3'$. The line of intersection of these two planes is parallel to the edges of the prism. Therefore, the points of intersection of the diagonals in all the plane cross sections of the prism are contained in the line

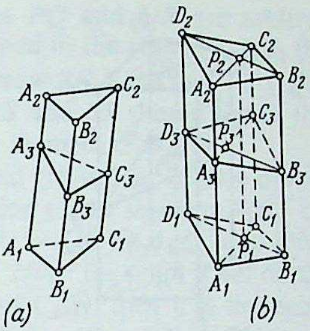


FIG. 23

which is parallel to the prism's edges. In Fig. 23*b* the points at which the diagonals of the bases intersect are labelled P_1 and P_2; the line P_1P_2 is parallel to the edges of the prism. The point P_3 is not contained in this line, and therefore the drawing is incorrect. This means that the depicted quadrilateral $A_3'B_3'C_3'D_3'$ is not a plane figure. Anyone with a trained eye (a painter, for example) would notice this at once without any auxiliary constructions.

The points of intersection of other pairs of lines may be chosen instead of the points P_1, P_2, P_3. For example, the point Q_1, which is the point of intersection of A_1B_1 and C_1D_1, and similar points Q_2 and Q_3 must lie on a line parallel to A_1A_2. The point R_1, at the intersection of A_1D_1 and B_1C_1, and similar points R_2 and R_3 must also lie on a line parallel to A_1A_2.

This property enables us both to check the correctness of the drawing and to construct the plane section of a quadrilateral prism. Choose the points A_3, B_3, and C_3 arbitrarily (Fig. 24). Construct the points P_1 and P_2 and draw the line P_1P_2 (parallel to A_1A_2). Draw the line A_3C_3 and find the point P_3 where A_3C_3 and P_1P_2 intersect. Draw the line B_3P_3. The point of intersection of B_3P_3 and D_1D_2 is the point sought.

We shall consider two more examples, but before doing so we shall explain why we have chosen them. This book deals with the representation of geometric objects rather than with the solution of problems of construction on paper. The second of these two topics is more complex than the first. Since it requires construction of plane cross sections in compliance with certain conditions, for example, the construction of a cross section of a polyhedron

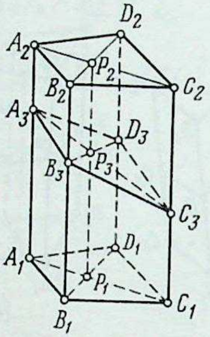


FIG. 24

by a plane that passes through three given points. This problem is more complicated especially if the given points lie on the edges. Therefore, we shall consider only the problem of how to 'represent *some* plane cross section of a polyhedron'.

When representing plane cross sections of polyhedrons, only two rules must be followed.

1. *If the planes* β' *and* γ' *intersect along the line* l' *and the plane* α' *intersects them along the lines* b' *and* c' *respectively, then* b' *and* c' *either meet in a point on* l' *or are parallel to* l'.

2. *If the planes* β' *and* γ' *are parallel and the plane* α' *intersects them along the lines* b' *and* c' *respectively, then* b' *and* c' *are parallel.*

Example 2. *Represent a plane cross section of a parallelepiped.*

Draw the images of the lines PQ and QR in the cross section. The point of their intersection lies on the line A_2B_2. In all other

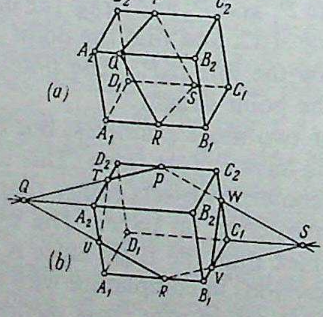


FIG. 25

respects, the lines PQ and QR are arbitrary. The point Q in Fig. 25*a* is contained in the segment A_2B_2 (however, this is not necessarily so). Draw $RS \parallel QP$ and $PS \parallel QR$. *Hint*: the point of intersection of RS and PS should lie on the line C_1D_1 (but not necessarily in the segment C_1D_1).

The lines PQ and QR in Fig. 25*b* are drawn so that the point Q lies on the extension of the segment A_2B_2. Label the points $T = PQ \cap A_2D_2$ and $U = QR \cap A_1A_2$, and join U to T. Draw $RS \parallel QP$ and $PS \parallel QR$. *Hint*: the point S lies on the line C_1D_1. If S lies on the extension of C_1D_1, then join the points $V = RS \cap B_1C_1$ and $W = PS \cap C_1C_2$. *Hint:* $VW \parallel TU$.

Example 3. *Represent a plane cross section of a quadrilateral pyramid* (Fig. 26).

Draw the segments PQ and QR arbitrarily. From the point R, the side of the cross section should be drawn across the back face. The planes of the back and front faces intersect along the line $S'X'$. The image of this line can be easily obtained by constructing the point $X = AB \cap CD$. The traces of the secant plane α' on the front and back faces should intersect along the line $S'X'$. Find the point $T = PQ \cap SX$ and draw the line TR. Mark the segment of this line RU, which lies within the triangle SCD. The line AB is the image of the line of the intersection of the front face and the lower base. This means that the trace of the secant plane on the front face ($P'Q'$) and the still unknown trace on the lower base $U'V'$ should intersect on the line $A'B'$.

17. Metric problems. Example 1. *Cut a cross section of a regular quadrilateral pyramid $S'A'B'C'D'$ with a plane perpendicular to the edge $S'A'$.*

The base of a regular quadrilateral pyramid is a square. It can be represented as an arbitrary parallelogram (Fig. 27). Moreover,

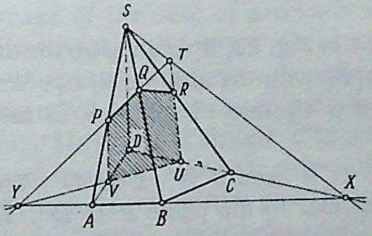


FIG. 26

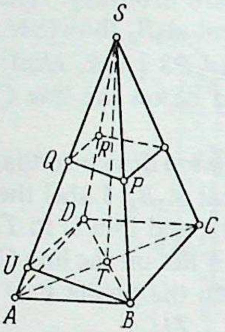


FIG. 27

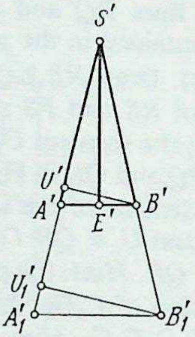


FIG. 28

a lateral edge may also be drawn in accordance with the Pohlke-Schwartz theorem, for example, $S'A'$.

Thus, a tetrahedron $S'A'B'C'$ is depicted arbitrarily. The Pohlke-Schwartz theorem states that a *given* tetrahedron can be represented arbitrarily. In the problem under consideration, however, the tetrahedron $S'A'B'C'$ is not defined because the length of the lateral edge (or, instead, the length of the altitude) is not given. Only if we add one more condition, for example

$$S'T' = 2 \cdot A'B',$$

is the problem fully defined.

Figure 28 shows the triangle $S'A'B'$ in its natural form, i.e. with no distortions. The point E' is the midpoint of the segment $A'B'$, $A'B' = a$, $S'E' = \dfrac{a\sqrt{17}}{2}$. The altitude $B'U'$ is dropped from the vertex B'.

Now we need to bring the point U' into the image. The ratio $\dfrac{S'U'}{U'A'}$ is invariant. We apply similarity transformation and construct $S'A_1' = SA$ in Fig. 28. By scaling up the drawing we obtain the triangle $S'A_1'B_1'$ with the altitude $B_1'U_1'$. We must still bring the dimension of $S'U_1'$ into Fig. 27, i.e. to measure off $SU \equiv S'U_1'$.

Now $S'A' \perp U'B'$ and $S'A' \perp U'D'$. Therefore, the line $S'A'$ is perpendicular to the plane $B'U'D'$. This plane can be translated. This translation replaces the images UB and UD of the

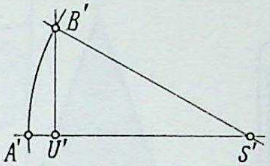


FIG. 29

traces of the plane by the parallel lines PQ and QR. This cross section can be completed according to the rules given in Sec. 16.

Note. If the length of the altitude or of the lateral edge is not given, even when it is known that the pyramid is a regular one, the problem is not defined. In such a case, only the converse problem can be solved. Draw BUD arbitrarily (see Fig. 27 again), and assume that $B'U'D' \perp S'A'$. The length of the altitude or of the lateral edge can now be determined. Draw the segment SA separately with the point U in it (Fig. 29). It is set that $\dfrac{S'U'}{U'A'} = \dfrac{SU}{UA}$ in Fig. 29. We may assume, for example, that $S'U' = SU$ and $S'A' = SA$. In Fig. 27 the line BU is not perpendicular to SA, while in the original (Fig. 29) we draw a line from the point U' perpendicular to the line $S'A'$. By swinging $S'A'$ around S' we get a point B'. Now we have the triangle $S'A'B'$ in its natural form of the lateral face of the pyramid (scaled up).

18. Solids of revolution. Cylinders. A cylinder is shown in Fig. 30. Both of its bases are represented as congruent ellipses. Figure 30*a* shows the images of two diameters of the upper (or the lower) base. The diameters are perpendicular and conjugate.

Figure 30*b* represents a sectional view of the cylinder with a cut made into the cylinder to form a dihedral angle equal to 90°.

Cones. The image of the base of a cone is an ellipse. In the classroom (even in school textbooks), a mistake is often made: extreme generatrices are used to represent the sides of the axial cross section and are assumed to touch the ellipse at the end points of the major axis (Fig. 31*a*). This is absurd. If tangents to the ellipse were drawn through the point S and the points of tangency A and B were joined, the line AB would not pass through the centre of the ellipse. Some 'draftsmen', nevertheless insist on drawing this

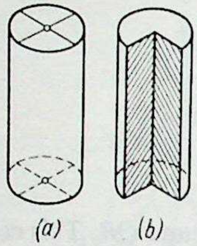
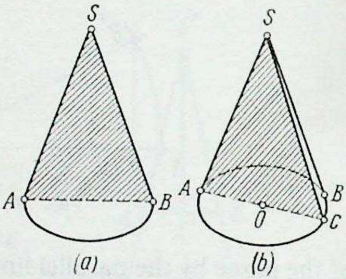

(a) (b)

FIG. 30



(a) (b)

FIG. 31

line through the centre. Figure 31*b* is correct. One of the points of tangency, say A, is joined to the centre O, and the point C, diametrically opposite A, is marked on the ellipse. The triangle SAC is the image of an axial cross section.

Figure 31*b* clearly shows that we see a bit more than one-half of the lateral surface of the cone. From what vantage-point will the cone appear to the observer as it is depicted in Fig. 31*b*? In fact, the observer would have to be quite far off and above the plane of the base (the light rays entering the eye form angles of about 30° with the plane of the base).

To draw the image of an axial cross section, it is not necessary to use one of the contour generatrices. Any two diametrically opposite generatrices will do. Figure 32 shows one more variant of drawing a cone with an axial cross section.

Spheres. Here we shall begin with a remark of a practical nature: the image of a sphere is normally an orthographic projection. This is explained by the fact that when a sphere is projected, the projecting lines form a circular cylinder touching the sphere (Fig. 33). The plane α' is perpendicular to the cylinder's

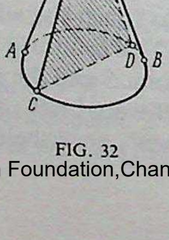


FIG. 32

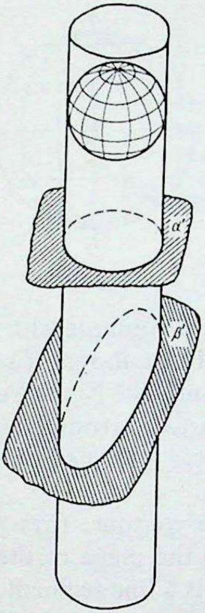


FIG. 33

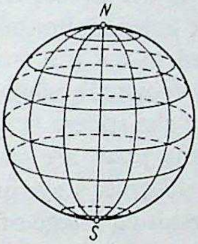


FIG. 34

generatrices while the plane β' is not; the cross section of the cylinder cut by α' is a circle, and that by β' is an elongated ellipse. In other words, the projection of the sphere onto α' is a circle while its projection onto β' is an elongated ellipse.

The image of a sphere whose contour is an elongated ellipse is not obvious and most people would say that the image does not resemble a sphere. Therefore, the plane α' is normally used rather than the plane β'.

Is it possible that a correctly constructed image of sphere (i.e. without mistakes) would still not be obvious? The answer, as seen from the example, is yes. In order for an image to be recognizable, it must necessarily be correct. That is not, however, sufficient. Other conditions will be discussed in Sec. 22.

Textbooks often contain the erroneous image of the Globe (Fig. 34). (In describing spheres, we will use conventional geographic terms, such as equator, meridian, and the North and South poles as if dealing with the Globe.)

In Fig. 34 the equator is shown as an ellipse whose lower part corresponds to the visible part of the equator. This means that

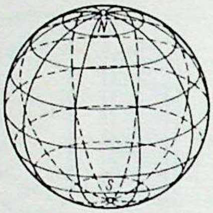


FIG. 35

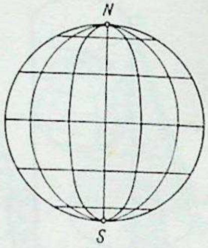


FIG. 36

rays projecting from the observer's eye are oblique with respect to the plane of the equator and pierce it from above. If so, the images of the poles cannot be on the contour. The North Pole should be situated lower (i.e. some of the area surrounding the pole should be depicted), while the South Pole is located on the invisible part of the sphere (Fig. 35).

When the poles are situated on the contour, rays projecting from the observer's eye are parallel to the plane of the equator, and the resulting image of the equator is a line segment (Fig. 36).

The correct image of a sphere is constructed as follows. The equator and meridians are drawn as ellipses, the meridians passing through two points *N* and *S*. We must still determine the relationship between the image of the equator and the position of the points *N* and *S*.

If the ellipse representing the equator were flattened into a line segment, then the points *N* and *S* would be situated on the contour. If the ellipse were extended, then the points *N* and *S* would be approaching each other. *The wider the ellipse, the lower the point N (and the higher the point S).* We shall explain this statement in more detail.

Imagine the original. Draw the equatorial cross section (circle *A′B′C′D′*), and draw the diameter *N′S′* (the Globe's axis)

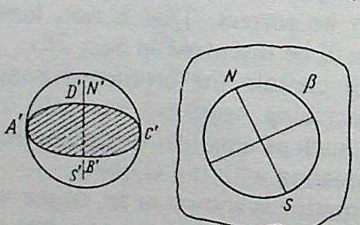


FIG. 37

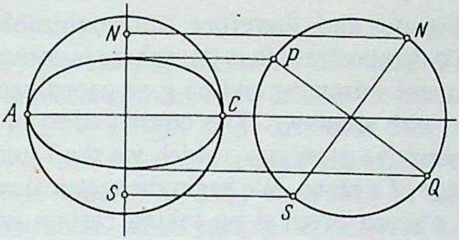


FIG. 38

perpendicular to this cross section. Imagine a plane perpendicular to $A'C'$ and passing (for convenience) outside the sphere (Fig. 37). Project the sphere together with its equatorial cross section and the axis $N'S'$ onto the plane β' orthogonally. The sphere is thus projected onto a circle, while the system ($A'B'C'D'$, $N'S'$) is projected onto two perpendicular diameters of this circle. If the figure on the left in Fig. 37 were revolved about $A'C'$, the circle on the plane β' would not move while the cross consisting of two perpendicular diameters would rotate around the centre.

Figure 38 suggests the answer to the question of how the image of the equator is linked to the images of the poles. The diameter PQ defines the minor axis of the ellipse while the diameter NS defines the position of the poles N and S. This drawing is explicit, and the reader will certainly understand it without explanation. The two following problems may be solved with the help of this drawing:

1. Assume the image of the equator is an ellipse. Find the poles.
2. Given the poles, construct the image of the equator.

Once the images of the equator and the poles are drawn, it is easy to draw a coordinate grid (meridians and parallels) on the sphere (Fig. 35). The meridians are represented by ellipses that pass through the points N and S. Parallel circles are drawn as similar ellipses touching upon the contour.

We are omitting details (i.e. the construction of the axes of all the ellipses) since the reader will be perfectly able to complete the drawing by himself.

Obviously, it may be completely unnecessary to show the coordinate grid on the image of the sphere. The original may be 'clear', i.e. with nothing on its surface. But in such a case, the image of the sphere would reduce to a circle (contour). The image would be ambiguous and, therefore, unrecognizable as the image of a sphere. To demonstrate that the sphere is convex, one should either superimpose a drawing on it (e.g., a coordinate grid) or supply the image with **shadows.** The construction of shadows is a branch of descriptive geometry, which we shall not discuss here.

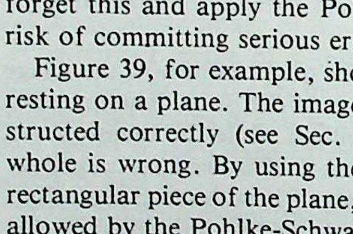
19. The image of a plane. We begin the metrical construction of the image of a given original by taking certain arbitrary steps. Each step, however, imposes certain constraints on the steps that follow. Finally, when we have constructed the image of a tetrahedron that is part of the original, our movements can no longer be arbitrary, and all remaining elements of the image must be constructed deliberately.

A plane (or rather 'a piece of a plane') is usually depicted as a parallelogram in school textbooks. The implication is that the original is a *rectangular* piece of a plane.

Representing a rectangular piece of the plane in the form of a parallelogram, we borrow a part of available freedom. If we later forget this and apply the Pohlke-Schwartz theorem, we run the risk of committing serious errors.

Figure 39, for example, shows a regular quadrilateral pyramid resting on a plane. The image $SABCD$, *taken separately*, is constructed correctly (see Sec. 13, example 2), but Fig. 39*a* as a whole is wrong. By using the parallelogram $KLMN$ to define a rectangular piece of the plane, we have partially spent the freedom allowed by the Pohlke-Schwartz theorem and may not define the square $A'B'C'D'$ by an arbitrary parallelogram. In Fig. 39*a* $\measuredangle NKL$ and $\measuredangle DAB$ are drawn as right angles. Since $AB \parallel KL$, the condition $AD \parallel KN$ should also be true. This is not so and Fig. 39*a* is wrong.

This mistake could have been avoided by drawing $AD \parallel KN$. If the problem concerns the properties of a pyramid resting on a



(a) (b)

FIG. 39

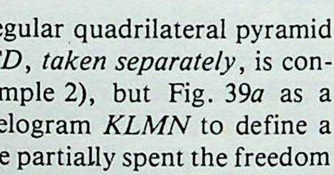


FIG. 40

plane, however, the rectangularity of the piece of the plane is unimportant. Therefore, there is no reason to introduce the constraint that $\measuredangle LKN$ is a right angle and thus complicate further constructions. For this reason, it is feasible to draw the plane as a piece with broken edges (Fig. 39*b*) since such an image does not involve any constraints. When casting the image of a quadrilateral pyramid placed on a plane, we have the same freedom of action as when we cast the image of a pyramid in empty space.

One other mistake is common. Figure 40 depicts a regular quadrilateral pyramid and a right circular cone set on the plane α, which has broken edges. If each of these solids has been set on separate 'footing', everything would be correct. But when we place them on a common plane, the base of each solid defines its metrics in that plane separately, and the metrics may not coincide.

Find a direction perpendicular to the line AB in Fig. 40. Since $ABCD$ is the image of a square, $A'D' \perp A'B'$. Draw the diameter $EF \parallel AB$ in the ellipse, and construct the diameter GH conjugate to EF. Thus, $G'H' \perp E'F'$, and the direction perpendicular to AB in the 'pyramid's metrics' has the line AD as its image, while in the 'cone's metrics' its image is the line GH. Since the lines GH and AD are not parallel, Fig. 40 is incorrect.

20. Inscribed and circumscribed solids. Exact drawing of inscribed and circumscribed solids requires complicated constructions. A drawing plays an auxiliary role in solving stereometric problems, and it is impractical to waste more effort on it than on the problem itself. Therefore, we advise the reader to make rough sketches. Familiarity with a few basic rules will help the student avoid certain mistakes.

1. *Inscribed and circumscribed spheres.* Spheres are more difficult to draw than polyhedrons, cylinders, or cones. Therefore, it is best to *start with drawing a sphere and then construct the remaining figures.*

2. *Spheres and cylinders.* If a sphere is inscribed in a cylinder (Fig. 41), it touches the lateral surface of the cylinder on its great circle, e.g. the equator. The point B lies on the equator, $AB = BC$, and $AC = NS$. All three ellipses (an equatorial cross section and the bases of the cylinder) are identical.

When a cylinder is inscribed in a sphere (Fig. 42), we note that the bases of the cylinder are identical parallel circles.

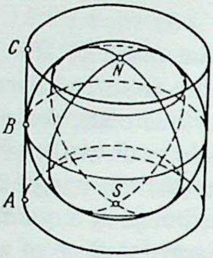


FIG. 41

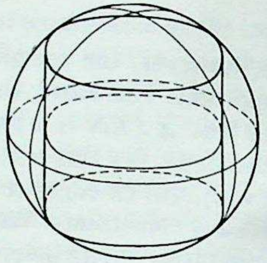


FIG. 42

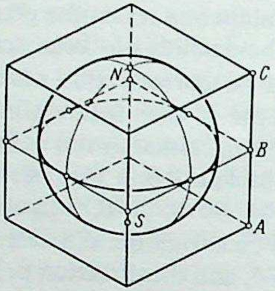


FIG. 43

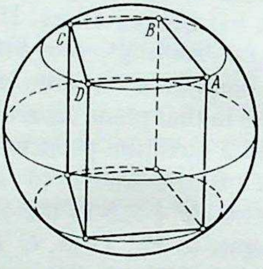


FIG. 44

3. *Spheres and prisms.* If a sphere is inscribed in a prism (Fig. 43), then the great circle (e.g. the equator) is inscribed in the midsection of the prism (i.e. the cross section cut by a plane that is parallel to the bases and that passes through the midpoint between them).

The drawing should be constructed as follows:

1. Draw a sphere.

2. Circumscribe a polygon about the equator, observing the conditions that specify this polygon. For example, if the polygon is a square (as in Fig. 43), its sides are parallel to the conjugate diameters of the ellipse.

3. Complete the prism in accordance with the following conditions: $AC = NS$, $AB = BC$.

To draw a prism inscribed in a sphere (Fig. 44), one should begin by inscribing a polygon in a parallel circle. Further construction should be clear without explanations.

4. *Spheres and pyramids.* If a sphere is inscribed in a pyramid (Fig. 45), then the points of tangency of the lateral faces are equidistant from the vertex of the pyramid. These points,

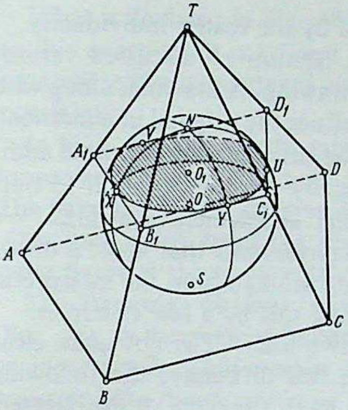


FIG. 45

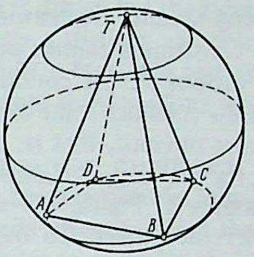


FIG. 46

therefore, lie on the same parallel circle, the plane of which is perpendicular to $T'O'$ (O' is the centre of the sphere and T' is the vertex of the pyramid).

If the plane of the base of the pyramid is parallel to the plane in which the points of tangency of the lateral faces are situated (this is true of regular pyramids), the order of construction is as follows:

(1) Draw a sphere with a coordinate grid on it (meridians and parallel circles).

(2) Circumscribe a polygon about a parallel circle (e.g., $A_1B_1C_1D_1$).

(3) Draw tangents from the tangential points X, Y, U, and V to the meridians that pass through these points. These lines meet in the point T which is the apex of the pyramid. Join T to the points A_1, B_1, C_1, and D_1.

(4) Complete the pyramid in accordance with the ratio

$$\frac{TA}{TA_1} = \frac{TS}{TO_1}$$

(S' is the South Pole and O is the centre of the parallel circle inscribed in $A_1'B_1'C_1'D_1'$).

If the plane of the base of the pyramid is not parallel to the plane of the given circle, the problem is much more complicated.

If a sphere is circumscribed about the pyramid (Fig. 46), then the base of the pyramid is inscribed in a parallel circle while its vertex may be any point on the sphere.

5. *Other cases.* To be considered by the reader individually.

21. Some drawing conventions. Section 15 discussed various conditions that may supplement a drawing. A drawing, along with these supplementary conditions, defines the original in a metrically precise manner. A drawing of the original without these additional constraints cannot be metrically precise*. Section 15 outlined various verbal conditions that supplement drawings. We shall now consider some *drawing conventions* that make a drawing more obvious, that is, eliminate ambiguity in the interpretation of a drawing. We shall illustrate this by a few examples.

Example 1. Figure 47*a* shows two lines. It is not quite clear whether they meet. To overcome this difficulty, the following convention is followed: if the lines meet, the point of the intersection of their images is marked by a small circle (Fig. 47*b*). If the lines do not intersect, the line which is farther from the observer is broken (Fig. 47*c*).

Example 2. When drawing a surface, we may consider it to be either opaque or semi-transparent. In the first case the lines hidden are not drawn. This is reasonable when the drawing is to be used as an illustration (e.g., when a material object is depicted). But it is not convenient in solid geometry where all lines must be visible. Therefore, surfaces are considered to be semi-transparent, and the lines passing behind them are drawn broken.

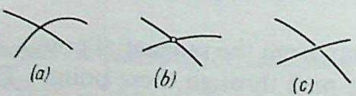


FIG. 47

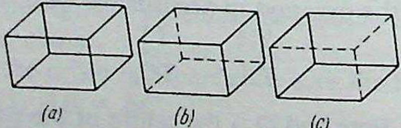


FIG. 48

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* We mean a drawing of a certain kind: an image on one plane obtained by parallel projections. There are other kinds of drawings (e.g., **Monge epure**, that is, orthogonal projections of an original on two mutually perpendicular planes) that define an original in a metrically precise manner with no additional constraints.

What is depicted in Fig. 48*a*? There is no unambiguous answer to this question. It may be a parallelepiped, or it may be twelve separate segments scattered in space (other interpretations are also possible).

In Figs. 48*b* and 48*c* the same solid is drawn but the figure is no longer ambiguous. It is now obviously a distinct parallelepiped. The parallelepiped is situated differently in Figs. 48*b* and 48*c*. The difference stems from the drawing conventions alone since the same figure is shown in both cases.

22. Drawing obvious images. As was mentioned above (Sec. 18), an image must be correct to be recognizable. Obviously, an incorrect drawing cannot be visually effective since the rules governing the construction of images are in accordance with the process of vision. An image must be more than correct, however. Figure 49 shows the photograph of a man with an outstretched hand. If we did not know that this is a photograph, we would certainly say: "This is impossible. This is not real. The painter made a mistake". But the camera never lies, and the image in Fig. 49, although not realistic, is correct.

Could a cube be drawn as shown in Fig. 50? According to the Pohlke-Schwartz theorem, the answer is yes. Yet most people would say that the figure depicted is not a cube. Why?

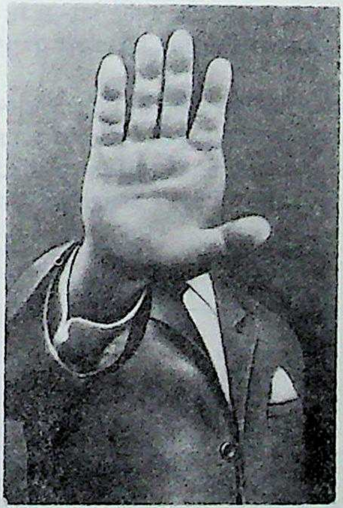

FIG. 49

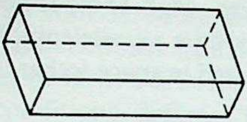

FIG. 50

Two more constraints are necessary to make an image obvious. First, the original must be shown from a *normal point of view*. We usually place an object right before our eyes when scrutinizing it. Furthermore, in the method of parallel projection, the object is placed *very far* from our eyes so that the rays of vision are perpendicular to the plane of projection. An image becomes more obvious if *projecting rays are inclined to the plane of projection at an angle of* 90° *or almost* 90°. The more this angle differs from 90°, the less obvious the image. We shall illustrate this statement by the following two examples.

Example 1. Let us return to Fig. 50. From what vantage-point does the cube look like this?

An unexperienced reader would answer: "It should be placed somewhat below the eyes and very far off to the right. One should squint one's eyes to the right".

A specialist would answer more precisely: "There are formulas in descriptive geometry that enable us to determine the projection of the cube, given its image. When the image in Fig. 50 was cast, the projecting rays were inclined at an angle of 14° to the plane of projection".

It is clear now why the drawing in Fig. 50 does not resemble a cube: we are not used to looking at cubes from such a viewpoint.

Example 2. As Sec. 18 points out, if the projection of the image is not orthogonal, then the sphere's image will not be recognizable (see Fig. 33).

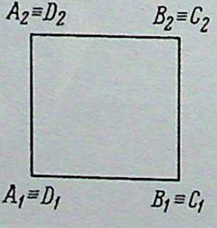

FIG. 51

The reader may wonder why the angle between the projecting rays and the plane of projections is only approximately 90°. Why do we not insist that this angle be exactly 90°?

The answer is simple. For the sake of obviousness, the following condition must be met: *essential details of the original must not obstruct each other*. To meet this requirement, it is possible to deviate from the exact value of 90°.

Figure 51 represents a cube. Although it is not recognizable as a cube, the image, however, is correct, and the angle is equal to 90°. The direction of projection is parallel to the edge $A_1'D_1'$, while the plane of projections is parallel to the face $A_1'B_1'B_2'A_2'$. This is the way a cube looks if viewed frontally from afar. This image is unclear because front vertices obstruct those located behind.

The fulfilment of the requirements mentioned above is enough to make an image more obvious.

CHAPTER 3

A Computation Method

23. Theory. So far, we have been considering the images of most elementary objects. But what if it is necessary to draw the image of something much more complicated? Specialists might be willing to pour over thick volumes on descriptive geometry looking for the answer, but the layman, who might also need to construct correct images, can use his time better otherwise.

The computation method solves this problem. Just as a hearing aid helps a person with defective hearing, the computation method helps a person unused to drawing. To draw an image, it is only necessary to compute the coordinates of the points of an image and plot them on a graph paper.

Fix cartesian orthogonal coordinate systems (X', Y', Z') in space and (ξ, η) in the plane of images (Fig. 52). A point $M'(x', y', z')$ in space has the point $M(\xi, \eta)$ on the plane as an image. This means that the coordinates of M are the functions of the coordinates of M':

$$\xi = F(x', y', z'), \qquad \eta = G(x', y', z').$$

It is worthwhile finding these functions. This problem can be solved with the help of the following theorem.

Theorem. *In parallel projection, the coordinates of an image point are linear functions of the original point, i.e.*

$$\xi = a_1x' + b_1y' + c_1z' + d_1, \qquad \eta = a_2x' + b_2y' + c_2z' + d_2. \quad (1)$$

Note. It does not matter whether both cartesian systems are orthogonal or not. The theorem is valid for any affine system. We shall, however, use only cartesian orthogonal systems. Interested readers will find the proof of the above theorem in Appendix 1. Those who prefer to take it for granted may omit that proof.

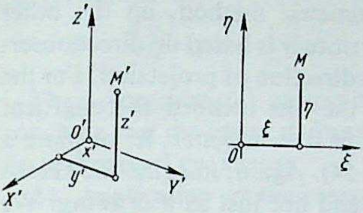

FIG. 52

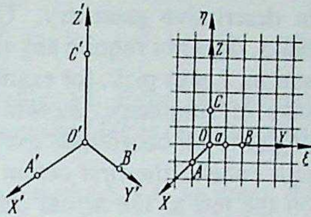

FIG. 53

The coefficients in formulas (1) may be chosen arbitrarily, but the resulting image may not satisfy us. It is better first to determine an image and then find respective coefficients.

When fixing the image we must observe the rules: we may fix the images of four unspecified points. The other points are determined from these four. The origin and the unit points are the easiest to choose.

Consider Fig. 53. The original is shown on the left (consider it the actual three-dimensional original rather than a drawing). On the right is an arbitrary image. The system (ξ, η) serves as a reference for the coordinates in the plane of projections. Since the position of the origin makes no difference, we shall identify its image O' with the point $\xi = 0$, $\eta = 0$. Then formulas (1) take the form

$$\xi = a_1x' + b_1y' + c_1z', \qquad \eta = a_2x' + b_2y' + c_2z'. \quad (2)$$

Now we know that (see Fig. 53):
the point $A'(1, 0, 0)$ is represented by the point $A(-a, -a)$;
the point $B'(0, 1, 0)$ is represented by the point $B(2a, 0)$;
the point $C'(0, 0, 1)$ is represented by the point $C(0, 2a)$.

By substituting these numbers into formulas (2), we determine all coefficients $a_1 = a_2 = -a$, $b_1 = c_2 = 2a$, $c_1 = b_2 = 0$. Thus, the image in Fig. 53 corresponds to the following formulas:

$$\xi = a(2y' - x'), \qquad \eta = a(2z' - x'). \quad (3)$$

The parameter a enables us to modify the size of the drawing.

The image of Fig. 53 may correspond to a non-orthogonal projection. But what if an orthogonal projection is needed? There are two solutions to this problem: analytical and geometric. The analytical solution, based on the application of formulas that define orthogonal projection*, is mainly of interest to specialists in descriptive geometry. The geometric method, on the other hand, does not require any theory since it is based on direct observation. Let us pick, for example, a direction of projection. For the sake of simplicity, we will pick the one inclined at congruent angles to all coordinate axes. To see it even better, let us place a unit cube in the first octant (Fig. 54). Again, imagine the sketch on the left as the actual original and not just as a drawing. We shall project in the direction of the diagonal $G'O'$ of the cube. The plane passing through O' and perpendicular to $G'O'$ (this is very important!) should be considered as the plane of projections. Obviously, the axes X', Y', and Z' project onto this plane so that the angles between the projections are congruent to each other (i.e., 120°), while the points A, B, and C are equidistant from O (Fig. 54, on the right). We place the axes (ξ, η) as shown in the drawing. The distance OA may be arbitrary because the image is not a direct projection but has also been subjected to similarity transformation.

Figure 54 shows that
the point $A'(1, 0, 0)$ is represented by the point $A\left(-\frac{a\sqrt{3}}{2}, -\frac{a}{2}\right)$;

the point $B'(0, 1, 0)$ by the point $B\left(\frac{a\sqrt{3}}{2}, -\frac{a}{2}\right)$;

the point $C'(0, 0, 1)$ by the point $C(0, a)$.

Determining the coefficients in formulas (2) from these conditions (using the method already explained), we obtain the follow-

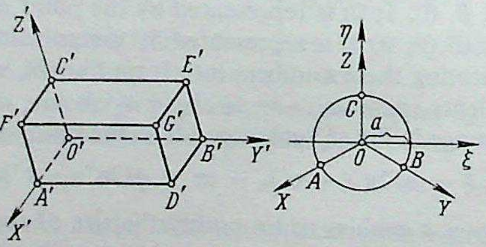

FIG. 54

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* See N. M. Beskin, *Methods of Drawing Objects*, Fizmatgiz, Moscow, 1963, p. 273 (in Russian).

ing formulas:

$$\xi = \frac{a\sqrt{3}}{2}(y' - x'), \qquad \eta = \frac{a}{2}[2z' - (x' + y')]. \tag{4}$$

Thus, formulas (4) correspond to Fig. 54. From this we can conclude that the *image defined by formulas* (4) *has been obtained by orthogonal projection*.

24. Application of the computation method. Let us apply the computation method to the construction of images much more complicated than those dealt with so far. Given *a sphere intersected by a circular cylinder. The radius of the cylinder is half the radius of the sphere. The generatrix of the cylinder passes through the centre of the sphere.* Represent *the sphere, the cylinder, and the line of their intersection* (this curve called the **Viviani curve**). Figure 55 represents the cross section of the sphere, cut by a plane that passes through the centre of the sphere and is perpendicular to the generatrices of the cylinder. Assume the radius of the sphere to be equal to unity. Then the equation of the sphere is

$$x'^2 + y'^2 + z'^2 = 1, \tag{5}$$

and the equation of the cylinder will be

$$\left(x' - \frac{1}{2}\right)^2 + y'^2 = \frac{1}{4}. \tag{6}$$

Note that the radius of the parallel circle may be found from the formula

$$r = \cos\varphi, \tag{7}$$

where φ is the latitude (here we are not defining the concepts of latitude and longitude since they are well-known geographical terms). The coordinates of each point contained in the sphere are

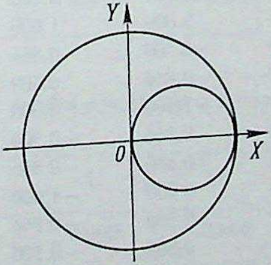

FIG. 55

expressed as follows:

$$\left.\begin{aligned} x' &= r\cos\theta \\ y' &= r\sin\theta \\ z' &= \sin\varphi \end{aligned}\right\} \qquad (8)$$

where φ is the latitude, θ is the longitude, and r is the radius of the parallel containing the point.

Construction of an image with the help of the computation method consists of the following three steps.

(1) Compute the coordinates of the points of the original;

(2) Compute the coordinates of the points of the image (according to the formulas (2));

(3) Plot the points on graph paper.

Step one. Draw parallel circles through each 30°, and compute their radii by formula (7):

$\varphi,°$	$r = \cos\varphi$
0	1.000
30	0.866
60	0.500
90	0.000

(the same values of r correspond to negative values of φ, i.e. southern latitude).

Now choose points on each parallel at longitudinal intervals of 30°, and compute their coordinates by formulas (8).

Table I′: parallel $\varphi = 0$, $r = 1$ (equator)

Point	$\theta,°$	x'	y'	z'
1	0	1.000	0.000	0.000
2	30	0.866	0.500	0.000
3	60	0.500	0.866	0.000
4	90	0.000	1.000	0.000
5	120	−0.500	0.866	0.000
6	150	−0.866	0.500	0.000
7	180	−1.000	0.000	0.000
8	210	−0.866	−0.500	0.000
9	240	−0.500	−0.866	0.000
10	270	0.000	−1.000	0.000
11	300	0.500	−0.866	0.000
12	330	0.866	−0.500	0.000

Table II′: parallel $\varphi = 30^\circ$, $r = 0.866$

Point	θ, °	x'	y'	z'
1	0	0.866	0.000	0.500
2	30	0.750	0.433	0.500
3	60	0.433	0.750	0.500
4	90	0.000	0.866	0.500
5	120	−0.433	0.750	0.500
6	150	−0.750	0.433	0.500
7	180	−0.866	0.000	0.500
8	210	−0.750	−0.433	0.500
9	240	−0.433	−0.750	0.500
10	270	0.000	−0.866	0.500
11	300	0.433	−0.750	0.500
12	330	0.750	−0.433	0.500

Table III′: parallel $\varphi = 60^\circ$, $r = 0.500$

Point	θ,°	x'	y'	z'
1	0	0.500	0.000	0.866
2	30	0.433	0.250	0.866
3	60	0.250	0.433	0.866
4	90	0.000	0.500	0.866
5	120	−0.250	0.433	0.866
6	150	−0.433	0.250	0.866
7	180	−0.500	0.000	0.866
8	210	−0.433	−0.250	0.866
9	240	−0.250	−0.433	0.866
10	270	0.000	−0.500	0.866
11	300	0.250	−0.433	0.866
12	330	0.433	−0.250	0.866

Table IV′: parallel $\varphi = 90^\circ$, $r = 0$ (North Pole)

Point	θ,°	x'	y'	z'
1	—	0.000	0.000	1.000

The tables for the parallel circles in the southern hemisphere differ only in the sign of z'. For example, the table for the parallel $\varphi = -30°$ looks as follows: x' and y' are the same as in Table II$'$, while $z = -0.500$.

Step two. Compute the tables containing the coordinates of the points of the image. Fix the image defined by formulas (4), and set $a = 100$ mm. To save space we shall simplify the tables, with the exception of Table I, by consolidating the pairs of parallel circles whose latitudes differ only in sign: ordinate η_1 refers to the northern parallel and η_2 to the southern.

Note that the two points, symmetric with respect to the plane of the equator, are different only in the sign of z'. If we first computed η_1 by the second of the two formulas in (4) and then, having replaced z' by $-z'$, computed z_2, we would obtain

$$\eta_1 - \eta_2 = 2az'$$

or (since $a = 100$ mm)

$$\eta_1 - \eta_2 = 200z'.$$

In each table $z' =$ const. This makes computation of η_2 very easy. For example, in Table II

$$\eta_2 = \eta_1 - 100.$$

Table I: parallel $\varphi = 0$ (equator)

Point	ξ	η
1	−86.6	−50.0
2	−31.7	−68.3
3	31.7	−68.3
4	86.6	−50.0
5	118.3	−18.3
6	118.3	18.3
7	86.6	50.0
8	31.7	68.3
9	−31.7	68.3
10	−86.6	50.0
11	−118.3	18.3
12	−118.3	−18.3

Table II: parallels $\varphi = \pm 30°$

Point	ξ	η_1	η_2
1	−75.0	6.7	−93.3
2	−27.5	−9.2	−109.2
3	27.5	−9.2	−109.2
4	75.0	6.7	−93.3
5	102.4	34.2	−65.8
6	102.4	65.8	−34.2
7	75.0	93.3	−6.7
8	27.5	109.2	9.2
9	−27.5	109.2	9.2
10	−75.0	93.3	−6.7
11	−102.4	65.8	−34.2
12	−102.4	34.2	−65.8

Table III: parallels $\varphi = \pm 60°$

Point	ξ	η_1	η_2
1	−43.3	61.6	−111.6
2	−15.8	52.4	−120.8
3	15.8	52.4	−120.8
4	43.3	61.6	−111.6
5	59.1	77.4	−95.8
6	59.1	95.8	−77.4
7	43.3	111.6	−61.6
8	15.8	120.8	−52.4
9	−15.8	120.8	−52.4
10	−43.3	111.6	−61.6
11	−59.1	95.8	−77.4
12	−59.1	77.4	−95.8

Table IV: parallels $\varphi = \pm 90°$ (North and South Poles)

Point	ξ	η_1	η_2
1	0.0	100.0	−100.0

At this stage we may move toward **step three** and draw the image of the sphere. The reader should make the drawing himself, plotting points in accordance with Tables I-IV. The following four notes will be very helpful.

Note one (very important!). Tables I-IV give the points of the parallels. But how are the images of the meridians to be drawn?

In the tables, all points with the same numbers have the same longitude. Therefore, *by joining the points that have the same numbers, we obtain the image of a meridian*. For example, all the points numbered 3 are contained in the meridian $\theta = 60^\circ$.

Tables I-IV, therefore, enable us to draw the image of the whole coordinate grid on the sphere.

Note two. The coordinates of the points in Tables I-IV are given in millimetres. The points should be plotted on graph paper in accordance with their coordinates without recomputation. The coordinates are given with accuracy to one decimal point because 0.1 mm is the highest accuracy of which a skilled draftsman is capable.

Parameter a in formulas (4) enables us to control the dimensions of a drawing. Note that if, to increase the scale, we multiplied all the coordinates in Tables I-IV by some factor significantly greater than unity, then the error would be much greater. Therefore, for a large drawing, the coordinates of original points should be computed with greater accuracy.

Note three. There is no need to compute the outline of a sphere from its points. It may be drawn with a compass. Its center is (0, 0), while the radius is equal to

$$100\sqrt{\frac{3}{2}}\ \text{mm} \cong 122.5\ \text{mm}.$$

Note four. If we are more concerned with drawing clear and recognizable figures than with solving geometric problems, the hidden lines that are obstructed from view should not be drawn because they make the drawing too complicated.

Let us consider the Viviani curve. It is obvious from equations (7) and (8) that the coordinates of each point of a unit sphere can be expressed as longitude and latitude in the following way:

$$\left.\begin{aligned} x' &= \cos\varphi\,\cos\theta, \\ y' &= \cos\varphi\,\sin\theta, \\ z' &= \sin\varphi. \end{aligned}\right\} \qquad (9)$$

The points of the sphere contained in the cylinder ought to comply with equations (9) and (6). Therefore, from these equations we obtain:

$$\left.\begin{aligned} x' &= \frac{1}{2}(1 + \cos 2\varphi),\\ y' &= \pm\frac{1}{2}\sin 2\varphi,\\ z' &= \sin\varphi. \end{aligned}\right\} \tag{10}$$

Equations (10) are the parametric equations of the Viviani curve. By assigning various values to φ, we obtain the points of this curve. If we set φ at intervals of 15° we obtain 24 points. Of course, only computations within the interval $0 \leq \varphi \leq 90°$ are necessary. Coordinates of other points are obtained symmetrically.

Note that the upper sign is sufficient for y'. The running point will pass through all points of the curve in any case, if φ changes from 0 to 360°. In fact, if formulas (10) with the upper sign were applied and if $\varphi_1 = 90° - \alpha$ and $\varphi_2 = 90° + \alpha$, then the same two points would be obtained as if formulas (10) were used with the double sign.

We compute the coordinates of the points of the image in accordance with formulas (4) with $a = 100$ mm.

To make the figure clearer, 'push' the cylinder slightly out of the sphere. Cut a cross section of the cylinder by planes slightly above and below the respective poles, e.g. $z' = \pm 1.1$. The coordinates of the points of the original may be computed with the following formulas:

$$\left.\begin{aligned} x' &= \frac{1}{2}(1 + \cos t),\\ y' &= \frac{1}{2}\sin t,\\ z' &= \pm 1.1. \end{aligned}\right\} \tag{11}$$

Now the reader can easily make the computations himself. The drawing in Fig. 56 has been constructed according to the results given in the tables (on a reduced scale). Hidden lines are not shown, with the exception of the Viviani curve.

When representing surfaces, obviousness of the image depends partly on the resolution power of the coordinate grid. Figure 57

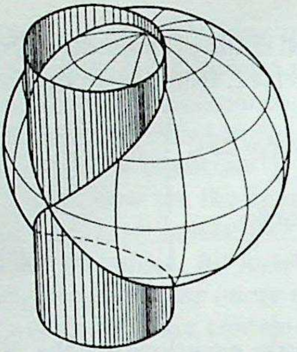

FIG. 56

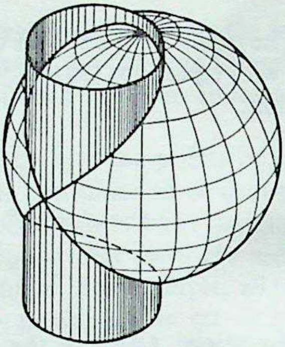

FIG. 57

Table V′: Viviani curve

Point	φ,°	x′	y′	z′
1	0	1.000	0.000	0.000
2	15	0.933	0.250	0.259
3	30	0.750	0.433	0.500
4	45	0.500	0.500	0.707
5	60	0.250	0.433	0.866
6	75	0.067	0.250	0.966
7	90	0.000	0.000	1.000
8	105	0.067	−0.250	0.966
9	120	0.250	−0.433	0.866
10	135	0.500	−0.500	0.707
11	150	0.750	−0.433	0.500
12	165	0.933	−0.250	0.259
13	180	1.000	0.000	0.000
14	195	0.933	0.250	−0.259
15	210	0.750	0.433	−0.500
16	225	0.500	0.500	−0.707
17	240	0.250	0.433	−0.866
18	255	0.067	0.250	−0.966
19	270	0.000	0.000	−1.000
20	285	0.067	−0.250	−0.966
21	300	0.250	−0.433	−0.866
22	315	0.500	−0.500	−0.707
23	330	0.750	−0.433	−0.500
24	345	0.933	−0.250	−0.259

differs from Fig. 56 only in that the parallels and the meridians have been drawn twice as dense: at intervals of 15°. Compare impressions from each of the drawings.

Table V: Viviani curve

Point	ξ	η	Point	ξ	η
1	− 86.6	− 50.0	13	− 86.6	− 50.0
2	− 59.1	− 33.3	14	− 59.1	− 85.0
3	− 27.5	− 9.2	15	− 27.5	− 109.2
4	0.0	20.7	16	0.0	− 120.7
5	15.8	52.4	17	15.8	− 120.7
6	15.8	80.8	18	15.8	− 112.4
7	0.0	100.0	19	0.0	− 100.0
8	− 27.5	105.8	20	− 27.5	− 87.4
9	− 59.1	95.8	21	− 59.1	− 77.4
10	− 86.6	70.7	22	− 86.6	− 70.7
11	− 102.4	34.2	23	− 102.4	− 65.8
12	− 102.4	− 8.3	24	− 102.4	− 60.0

APPENDIX 1

Expression of the Coordinates of the Image Points Using the Coordinates of the Original Points

25. A characteristic property of a linear homogeneous function. A characteristic property of an object or a set of objects is a property possessed by that object or set *alone*, which distinguishes it from all other objects or sets. For example, the number 2 is an even prime. This is its characteristic property.

The function $f(x) = ax + b$ is a **linear** function. When $b = 0$, the function is a **linear homogeneous** function.

If the equation

$$f(x_1 + x_2) = f(x_1) + f(x_2) \tag{12}$$

is an *identity*, that is, *correct for any two values of the argument* x_1 *and* x_2, then the function $f(x)$ is **additive.** Clearly, a linear homogeneous function is additive. In fact, if $f(x) = ax$, then $f(x_1 + x_2) = a(x_1 + x_2) = ax_1 + ax_2 = f(x_1) + f(x_2)$. Do any other additive functions exist? The answer for continuous functions is no. The linear homogeneous function is the only additive function. Additivity is its characteristic property. The proof follows.

Lemma 1. *If $f(x)$ is a continuous function* and for any two values of its argument $f(x_1 + x_2) = f(x_1) + f(x_2)$, then*

$$f(x) = ax.$$

Note. Continuity is not obligatory since the lemma is correct for more general conditions. We set this condition to simplify the proof.

Proof. Let α be any nonzero real number. Since $f(2\alpha) = f(\alpha + \alpha) = f(\alpha) + f(\alpha) = 2f(\alpha)$, $f(3\alpha) = f(2\alpha + \alpha) = f(2\alpha) +$

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* On the continuous function see N. Ya. Vilenkin and S. I. Schwartzburd, *Mathematical Analysis*, Moscow, 1969, Chapter III, § 3 (in Russian).

$f(\alpha) + 2f(\alpha) + f(\alpha) = 3f(\alpha)$ and so on, by the induction $f(n\alpha) = nf(\alpha)$. If we set $\alpha = 1, f(1) = a$, then $f(n) = an$, that is, the lemma is already proved for natural values of the argument. Now set $\alpha = 1/q$, where q is a natural number. As was proved above, $a = f(1) = f(q\alpha) = qf(\alpha) = qf\left(\frac{1}{q}\right)$. Therefore,

$$f\left(\frac{1}{q}\right) = a\frac{1}{q}.$$

Let p be a natural number as well. Then

$$f\left(\frac{p}{q}\right) = pf\left(\frac{1}{q}\right) = a\frac{p}{q}.$$

The lemma has thus been proved for all positive rational values of the argument. If x is an irrational (positive) number, then it may be presented as the limit of a sequence of rational numbers:

$$r_1, r_2, \ldots, r_n, \ldots, \lim_{n \to \infty} r_n = x.$$

Since the function in question is continuous,

$$f(x) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} (ar_n) = ax.$$

To extend the lemma to nonpositive values of the argument, note that $f(0) = f(0 + 0) = 2f(0)$, therefore

$$f(0) = 0.$$

Let $x = -\alpha$, with $\alpha > 0$. We know that $f(\alpha - \alpha) = f(0) = 0$, but we also know that $f(\alpha - \alpha) = f[\alpha + (-\alpha)] = f(\alpha) + f(-\alpha)$. Therefore,

$$f(x) = f(-\alpha) = -f(\alpha) = -a\alpha = a(-\alpha) = ax.$$

The lemma has thus been proved for all real numbers and may readily be extended to functions of several variables.

Lemma 2. *A function of several variables that is continuous with respect to each argument and that possesses the property of additivity, i.e.*

$$F(x_1 + x_2, y_1 + y_2) = F(x_1, y_1) + F(x_2, y_2)^*, \qquad (13)$$

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* For the sake of brevity, we use the symbol of a function of two variables but the formulation of the lemma and the proof are valid for functions of any number of variables.

is a linear homogeneous function, *i.e.*

$$F(x, y) = ax + by.$$

Proof. If $y = 0$, then the function $F(x, y)$ becomes a function of a single variable $F(x, 0)$. If $y_1 = y_2 = 0$ in formula (13), then $F(x_1 + x_2, 0) = F(x_1, 0) + F(x_2, 0)$. Therefore, according to Lemma 1, $F(x, 0) = ax$. The equation $F(0, y) = by$ may be proved in the same way.

The property in (13) gives the equation

$$F(x, y) = F(x, 0) + F(0, y) = ax + by$$

which was to be proved.

26. Formulas for the coordinates of the points of an image. Lemma 3. *If the points* $O'(0, 0, 0)$, $M_1'(x_1', y_1', z_1')$ and $M_2'(x_2', y_2', z_2')$ *are represented by the respective image points* $O(0, 0)$, $M_1(\xi_1, \eta_1)$ *and* $M_2(\xi_2, \eta_2)$, *then the point* $M_3'(x_1' + x_2', y_1' + y_2', z_1' + z_2')$ *is represented by the point* $M_3(\xi_1 + \xi_2, \eta_1 + \eta_2)$.

If the points O', M_1', and M_2' are not contained in the same line, then $O'M_1'M_2'M_3'$ is a parallelogram with O' and M_3' as its opposite vertices. The image of a parallelogram is a parallelogram. With the addition of the point $M_3(\xi_1 + \xi_2, \eta_1 + \eta_2)$, the triangle OM_1M_2 becomes a parallelogram (proof: the midpoint of the segment OM_3 coincides with the midpoint of the segment M_1M_2). Hence, M_3 is the image of the point M_3'.

If the points O', M_1', and M_2' are contained in the same line, then

$$\frac{x_1'}{x_2'} = \frac{y_1'}{y_2'} = \frac{z_1'}{z_2'}. \tag{14}$$

By adding unity to all parts of equations (14), we get

$$\frac{x_1' + x_2'}{x_2'} = \frac{y_1' + y_2'}{y_2'} = \frac{z_1' + z_2'}{z_2'}, \tag{15}$$

i.e. the *point* $M_3'(x_1' + x_2', y_1' + y_2', z_1' + z_2')$ *lies on* the same line.

We know that the points O, M_1, and M_2 are the images of the points O', M_1', and M_2'. This means, firstly, that the points O, M_1, and M_2 lie on the same line, i.e.

$$\frac{\xi_1}{\xi_2} = \frac{\eta_1}{\eta_2}, \tag{16}$$

and, secondly, that $(M_1M_2O) = (M'_1M'_2O')$. Since

$$(M_1M_2O) = \frac{M_1O}{OM_2} = -\frac{\xi_1}{\xi_2}, \qquad (M'_1M'_2O') = \frac{M'_1O'}{O'M'_2} = -\frac{x'_1}{x'_2},$$

we have

$$\frac{\xi_1}{\xi_2} = \frac{x'_1}{x'_2}. \tag{17}$$

From (16), it follows that

$$\frac{\xi_1 + \xi_2}{\xi_2} = \frac{\eta_1 + \eta_2}{\eta_2}.$$

This means that the point $M_3(\xi_1 + \xi_2, \eta_1 + \eta_2)$ lies on the line OM_1M_2. We shall prove that this point divides the segment M_1M_2 in the same ratio as the point M'_3 divides the segment $M'_1M'_2$:

$$\frac{M_1M_3}{M_3M_2} = \frac{(\xi_1 + \xi_2) - \xi_1}{\xi_2 - (\xi_1 + \xi_2)} = -\frac{\xi_2}{\xi_1},$$

$$\frac{M'_1M'_3}{M'_3M'_2} = \frac{(x'_1 + x'_2) - x'_1}{x'_2 - (x'_1 + x'_2)} = -\frac{x'_2}{x'_1}.$$

From (17), it follows that

$$\frac{M_1M_3}{M_3M_2} = \frac{M'_1M'_3}{M'_3M'_2}.$$

This means that the point M_3 is the image of the point M'_3. This proves Lemma 3.

If the image of space is cast onto a plane by parallel projection, then each point $M'(x', y', z')$ of space is associated with a point on the plane $M(\xi, \eta)$. This means that the coordinates of the point M are functions of the coordinates of the point M':

$$\xi = F(x', y', z'), \qquad \eta = G(x', y', z'). \tag{18}$$

These functions are continuous (both with respect to each argument and to the totality of the arguments). It is rather easy to give a precise proof of the continuity, but we prefer to appeal to the reader's intuition: the points of space situated very close to each other obviously have their image points also situated close to each other.*

Now Lemma 3; we shall confine ourselves to the first formula for the points M'_1, M'_2, and M'_3:

---

* The converse statement is not true!

$$\xi_1 = F(x_1', y_1', z_1'),$$

$$\xi_2 = F(x_2, y_2', z_2'),$$

$$\xi_1 + \xi_2 = F(x_1' + x_2', y_1' + y_2', z_1' + z_2').$$

From this it is clear that

$$F(x_1' + x_2', y_1' + y_2', z_1' + z_2') = F(x_1', y_1', z_1') + F(x_2', y_2', z_2'). \tag{19}$$

The following expression stems from identity (19) if Lemma 2 is taken into account:

$$F(x', y', z') = a_1 x' + b_1 y' + c_1 z'.$$

Similar statements also refer to the function $G(x', y', z')$:

$$G(x', y', z') = a_2 x' + b_2 y' + c_2 z'.$$

Now let us free ourselves from the constraint that an origin must have an origin as its image. Suppose that a point $O'(0, 0, 0)$ is represented by a point $O(d_1, d_2)$. Introduce one more coordinate system (ξ^*, η^*) in the plane of images, in addition to the coordinate system (ξ, η), and define it by the following formulas:

$$\xi^* = \xi - d_1, \quad \eta^* = \eta - d_2. \tag{20}$$

The image of the point $O'(0, 0, 0)$ has coordinates $\xi = d_1$ and $\eta = d_2$, while its coordinates in the new system are $\xi^* = 0$, $\eta^* = 0$. According to the proof, the new coordinates of each point in the image are expressed as follows:

$$\xi^* = a_1 x' + b_1 y' + c_1 z', \qquad \eta^* = a_2 x' + b_2 y' + c_2 z',$$

while the old coordinates are expressed as:

$$\begin{aligned} \xi &= a_1 x' + b_1 y' + c_1 z' + d_1, \\ \eta &= a_2 x' + b_2 y' + c_2 z' + d_2. \end{aligned} \tag{21}$$

This is what we had to prove.

As seen from the above analysis, the orthogonality of both systems is unimportant.

The converse theorem is also valid: *if the coefficients a_2, b_2, and c_2 are not proportional to the coefficients a_1, b_1, and c_1, then formulas* (21) *define the image of space onto the plane by parallel projection.*

Since the focus of this book is geometry, we shall not be concerned here with analytical geometry.

APPENDIX 2

The Ellipse

27. Uniform compression. The easiest way to develop a theory of the ellipse is by the application of **affine transformations.** The study of affine transformations, however, is such an important theme in itself (even more important than that of this book), that it would not be proper to expound it as an auxiliary topic in an appendix. We advise the reader, therefore, if he is interested in a serious study of affine transformations and the theory of ellipses, to familiarize himself with the literature (e.g., I. M. Yaglom and B. G. Ashkinuse, *The Ideas and Methods of Affine and Projective Geometry*, Part 1. *Affine geometry*. Moscow, 1962, §§ 1-17, in Russian). Here, as a *reference only*, we shall explain the properties of the ellipse that are necessary to understand how to represent the image of a circle.

Uniform compression involves the following transformation of a plane. A line called the **axis of compression** is chosen, and each point M of the plane moves along the perpendicular to the axis of compression to a new position M' (Fig. 58). In addition,

$$M_0M' = \lambda \cdot M_0M, \tag{22}$$

M_0 is the projection of the points M and M' on the axis of compression and λ is a constant here, i.e. it has the same value for all points of the plane. If $M \to M'$ (this notation means: 'point M moves to point M'' or 'point M corresponds to point M'') and $N \to N'$, then

$$M_0M' = \lambda \cdot M_0M,$$
$$N_0N' = \lambda \cdot N_0N,$$

λ is called the **coefficient of compression.** We assume that $\lambda > 0$. It is also possible to define compression using the negative coeffi-

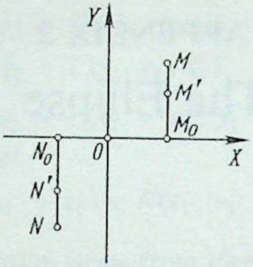

FIG. 58

cient (the corresponding points would then lie on both sides of the axis of compression), but we shall do without this operation.

Let us mention some properties of uniform compression.

1. Each point of the axis of compression remains stationary (i.e. it corresponds to itself).

2. If $\lambda < 1$, then all points (that do not belong to the axis of compression) approach the axis. If $\lambda > 1$, then all points move away from the axis. In this case, transformation (22) would more properly be called *extension* rather than compression. But uniform terms are preferable in mathematics, however much they contradict normal usage. Therefore we shall always use the term 'compression', even if $\lambda > 1$.

If $\lambda = 1$, then each point remains stationary, and the axis of compression is undefined. This case is hardly interesting except as an example of a particular transformation called **identical transformation.**

3. A straight line becomes a straight line. The transformation of a line (and, in general, of any figure) means the transformation of its each point. If a line intersects the axis of compression, then the corresponding line also intersects it (at the same point) (Fig. 59*a*). If line a forms angle α with the axis of compression and line a' forms angle α', then

$$\tan\alpha' = \lambda\tan\alpha.$$

A line parallel to the axis of compression becomes another line, which is also parallel to the axis of compression (Fig. 59*b*). A line perpendicular to the axis of compression becomes itself (although its points move along the line).

4. Parallel lines become parallel lines. This is clear from the foregoing property.

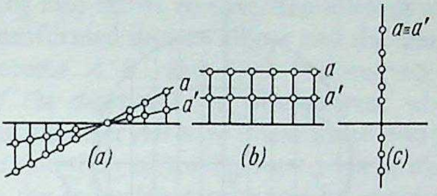

FIG. 59

5. If the three points A, B, and C are contained in a line, then their division ratio is invariant with respect to uniform compression, i.e.

$$\frac{A'C'}{C'B'} = \frac{AC}{CB}. \tag{23}$$

(This may be easily proved. Look at Fig. 59.) In particular, the midpoint of the segment becomes midpoint of the line.

6. Two mutually perpendicular directions do not, as a rule, become two mutually perpendicular ones. But there are cases when $a \perp b$ and $a' \perp b'$. We shall list these cases below.

(1) If $\lambda \neq 1$, then there exists the *unique* pair of mutually perpendicular directions that remain perpendicular after compression: the one parallel to the axis of compression and the one perpendicular to it. They are called the **principal directions** of compression.

(2) If $\lambda = 1$, then any pair of mutually perpendicular lines remains perpendicular after compression, i.e. the principal directions remain undefined (all directions are principal).

7. The transformation inverse with respect to a uniform compression is also a uniform compression (toward the same axis).

Two transformations are called **mutually inverse** if the first one carries each point M of the plane to a new position M', while the second one returns each point M' to position M. In other words, if a plane is subjected to two successive mutually inverse transformations, then all its points remain at their initial positions. Property (7) follows from formula (22), which gives

$$M_0M = \frac{1}{\lambda} M_0M'.$$

It is clear that the formulas look alike, and the coefficients of these mutually inverse compressions are λ and $1/\lambda$.

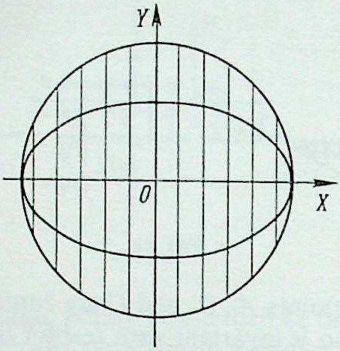

FIG. 60

8. Let the axis X be the axis of compression (Fig. 58). Now we express the uniform compression analytically. If point $M(x, y)$ moves into point $M'(x', y')$, then

$$x' = x, \qquad y' = \lambda y. \tag{24}$$

28. The definition of an ellipse. *An ellipse is a curve produced by uniform compression of a circle toward its diameter* (Fig. 60).

Note 1. There are different ways to define an ellipse. The best way is to define it as a curve obtained from a circle by any affine transformation. This is possible if the affine transformation is defined. Uniform compression is a special example of an affine transformation.

Note 2. The coefficient of the compression mentioned in the definition may have any positive value. In Fig. 60, $\lambda < 1$. If $\lambda = 1$, then a circle would be obtained. Therefore, a circle is a particular kind of ellipse.

Note 3. We can distinguish between a 'periphery' and a 'circle'. The term 'periphery' refers only to the contour of a circle. The term 'ellipse', which can mean both a part of a plane and its contour, is less exact.*

29. Some properties of ellipses. The diameters AB and CD in Fig. 61 are two mutually perpendicular diameters of a circle. Each of them divides in half the chords parallel to the other. The drawing shows the chords parallel to AB and marks their midpoints, which lie on CD.

---

* This is a specific feature of Russian. In English the term 'circle' may mean both a part of a plane and its contour. — *Tr.*

The drawing also shows compression towards a diameter. The circle has transformed into an ellipse and the diameters AB and CD have become $A'B'$ and $C'D'$ respectively. The *perpendicularity of the diameters has disappeared*, while the chords parallel to the diameter AB have transformed into parallel chords of the ellipse and their midpoints have become midpoints again. We arrive at the following conclusions by comparing the ellipse with the circle.

(1) *The midpoints of the parallel chords of the ellipse are contained in the same line.*

The locus of the midpoints of the parallel chords is called the **diameter** of the ellipse **conjugate** to the chords.

(2) We may consider the chords to be parallel to this diameter. The diameter conjugate to them is contained in the first family of chords. Hence, the conjugation is a mutual property.

Two diameters of an ellipse are called conjugate if each of them divides in half the chords parallel to the other.

(3) *All diameters of an ellipse pass through a single point, which is called the centre of the ellipse. The centre of an ellipse is the centre of symmetry.*

(4) *The tangents to an ellipse passing through the ends of the same diameter are parallel to the conjugate diameter.*

Mutually perpendicular diameters of a circle under compression become conjugate diameters of the ellipse. The new diameters are not generally perpendicular. The only exception concerns perpendicular diameters that had principal directions. This means that one diameter lies on the axis of compression while the other is perpendicular to it. Thus:

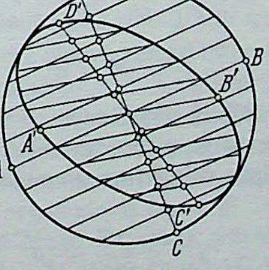

FIG. 61

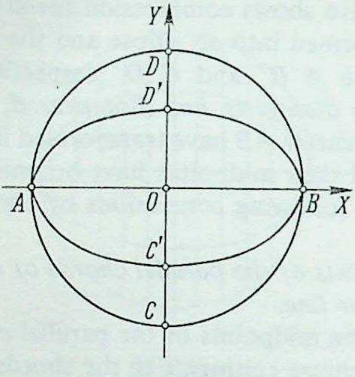

FIG. 62

(5) *An ellipse that is distinct from the circle possesses a single pair of mutually.perpendicular conjugate diameters.*

These diameters are called the **axes of the ellipse** and are its **axes of symmetry**. In Fig. 62, $\lambda < 1$. The line AB is the **major axis.** The line $C'D'$ is the **minor axis.** The major axis is traditionally designated as $2a$ and the minor one as $2b$. $OB = a$ is the semi-major axis, $OD' = b$ the semi-minor axis. The ends of the axes are called the **vertices** of the ellipse.

It is clear now that if $\lambda > 1$, then the line AB becomes the minor axis while the line $C'D'$ becomes the major one. Therefore, we shall consider only the case of $\lambda < 1$.

It should be obvious from the above that $b = \lambda a$, i.e.

$$\lambda = \frac{b}{a}\,. \tag{25}$$

(6) Let us derive the equation of an ellipse by placing the coordinate axes as shown in Fig. 60. The equation for a circle is

$$x^2 + y^2 = a^2. \tag{26}$$

We express the running coordinates of the points (x, y) of the circle in terms of the running coordinates of their respective points (x', y') on the ellipse using formulas (24). Then equation (26) becomes

$$x'^2 + \frac{y'^2}{\lambda^2} = a^2.$$

Replace λ by its expression in (25)

$$x'^2 + \frac{y'^2}{\left(\frac{a}{b}\right)^2} = a^2$$

or simply

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

The prime signs have been introduced here to distinguish the coordinates of the points of the ellipse from those of the circle.

If we consider the ellipse separately, forgetting the circle, then the primes are unnecessary:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{27}$$

(7) The ratio of semi-axes b/a lies within the range

$$0 < \frac{b}{a} \leq 1. \tag{28}$$

This ratio defines the shape of the ellipse. If b/a is small, then the ellipse is greatly elongated. If b/a increases, then the ellipse becomes 'rounder'. If $b/a = 1$, then the ellipse becomes a circle.

We have excluded the possibility of $b/a = 0$, i.e. $b = 0$, because the ellipse then degenerates into a double segment. Is the double segment then actually an ellipse? We may consider it to be, in which case uniform compression with $\lambda = 0$ would be valid, and some properties of uniform compression, which were discussed above, would need to be reconsidered.

30. The ellipse as the projection of a circle. Theorem. *If the two planes α and β are not perpendicular to each other and a circle is contained in α, then the orthogonal projection of the circle onto β is an ellipse.*

Proof. Suppose that the planes α and β intersect (Fig. 63). Draw the diameter $A'B'$ parallel to β in the circle (i.e. parallel to the line of intersection of the planes α and β). The projection of this diameter is the segment AB, which is congruent to $A'B'$ and also parallel to the line of intersection of the planes. Designate the length of each of these segments as $2a$: $A'B' = AB = 2a$. Consider now any chord $M'N'$ of the circle perpendicular to $A'B'$. The point P' is the midpoint of the chord $M'N'$. The projection

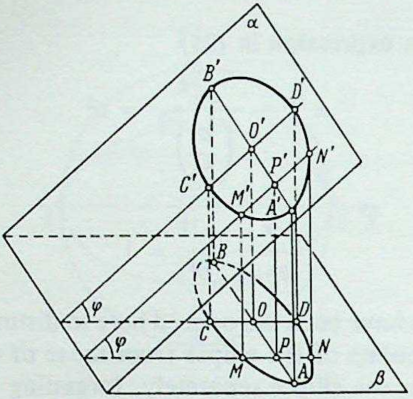

FIG. 63

of the semi-chord $P'N'$ is the segment PN, which is perpendicular to AB (remember the theorem about the three perpendiculars). The point P occupies the same position on the line AB as the point P' on the line $A'B'$, i.e. $OP = O'P'$. The length of the projection of $P'N'$ may be determined from the well-known formula $PN = P'N' \cdot \cos\varphi$, where φ is a linear angle of the dihedral angle between the planes α and β.

The diameter $A'B'$ has been projected onto β in its actual size. All semi-chords $P'N'$ have been carried 'into their own positions'. If they had retained their original dimensions, then the circle in the plane β would have been the same as that in the plane α. In fact, these semi-chords reduce in size when carried into the plane β: *their lengths are multiplied by the same coefficient* (Attention! This is the main point of the proof), which is equal to the cosine of the angle between the two planes. Therefore, the projection in question results from uniform compression of the circle toward its diameter, i.e. an ellipse.

We have assumed that the planes α and β intersect. Parallel planes represent an even easier case: the projection of a circle is also a circle. This completes the proof.

One more remark. Among the semi-chords of the circle perpendicular to $A'B'$, there is a radius $O'D' = a$. Obviously, the projection of this radius is the semi-minor axis b of the ellipse. This means that $b = a \cdot \cos\varphi$.

The theorem we set out to prove may, therefore, be completed as follows: '*The ratio of the semi-axes of the ellipse is equal to the*

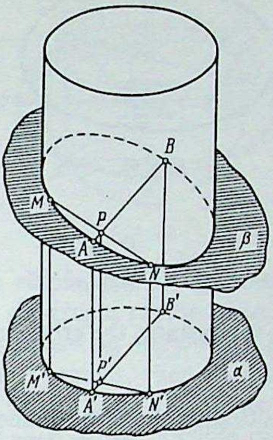

FIG. 64

cosine of the angle between the planes α *and* β', i.e.

$$\frac{b}{a} = \cos\varphi. \tag{29}$$

If α and β are perpendicular, then the projection of the circle is a double segment.

31. The cross sections of a circular cylinder. The following **theorem** is valid: *The cross section of a circular cylinder by a plane that is not parallel to the cylinder's generatrices is an ellipse.*

Note. An *infinite* cylinder is implied above, i.e. an infinite tube without bases.

We are not going to expound the proof, because it nearly coincides with the one before it. On one point the proofs differ, however. Figure 64 shows the cross section of a circular cylinder by the plane β. It is not yet clear whether this cross section is an ellipse. A normal cross section (by the plane α) of the cylinder is also shown in Fig. 64. This is a circle. Draw the diameter $A'B'$ of the circle parallel to the plane β (such a diameter is unique) and 'raise' its ends along the generatrices of the cylinder up to β. We thus obtain the chord AB, which is equal to $A'B'$. Let $P'N'$ be a semi-chord of the circle which is perpendicular to $A'B'$. By raising this chord to the plane β, we may obtain the segment PN. Thus, this proof differs from the proof preceding it: the segment PN may be obtained *by division by* $\cos\varphi$ *rather than by*

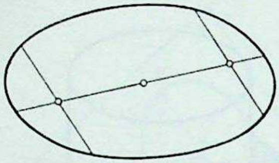

FIG. 65

multiplication:

$$PN = \frac{P'N'}{\cos\varphi}.$$

Since the factor $\frac{1}{\cos\varphi}$ is constant, the cross section by the plane β is an ellipse.

Unlike the former situation, here the diameter of the circle that retains its value

$$AB = A'B'$$

is the *minor rather than the semi-major axis* of the ellipse. But formula (29) is also valid in this case.

32. Some constructions connected with the ellipse. 1. Given an ellipse (the ellipse is drawn), find its centre. We draw two parallel chords (Fig. 65). Divide each chord in half. Join their midpoints to obtain a diameter. Divide the diameter in half. The midpoint of the diameter is the centre of the ellipse.

2. Given a diameter, construct a diameter conjugate to it. We draw a chord parallel to the given diameter. Divide the diameter and the chord in half. Join the midpoint of the diameter (the centre of the ellipse) and the midpoint of the chord to obtain the diameter conjugate to the given diameter.

3. Given a point M on an ellipse, construct a line tangent to the ellipse at the given point. Join point M and the centre O of the ellipse. Construct the diameter conjugate to the given diameter OM. Pass a line through the point M parallel to this conjugate diameter to obtain the desired tangent.

4. Construct the axes of an ellipse. If point M describes the quarter of the ellipse from the end of the major axis to the end of the minor axis, then its radius-vector OM changes continuously and equals each value between a and b only *once*. When sweeping other quarters of the ellipse, the radius-vector equals those values once in each quarter.

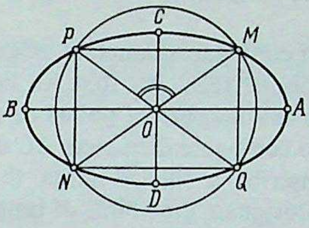

FIG. 66

Two points of an ellipse that are symmetrical with respect to one of its axes have congruent radius-vectors. Conversely, if two points of an ellipse situated on one side of the major (minor) axis have congruent radius-vectors, then they are symmetrical with respect to the minor (major) axis (this stems from the fact that the radius-vector equals each of the possible values *only once*).

This gives rise to the following method of axis construction (Fig. 66). Construct a circle, whose centre coincides with the centre of an ellipse O, and whose radius r is greater than b and less than a:

$$b < r < a.$$

This circle intersects the ellipse at the four points M, P, N, and Q. These points have congruent radius-vectors and, hence, are symmetrical with respect to the axes of the ellipse. Therefore, it remains to reconstruct the axes of symmetry of the four points M, P, N, and Q. To this end, one must

either draw the bisectors of the angles formed by the lines MN and PQ.

or draw the centre lines of the quadrilateral $MPNQ$.

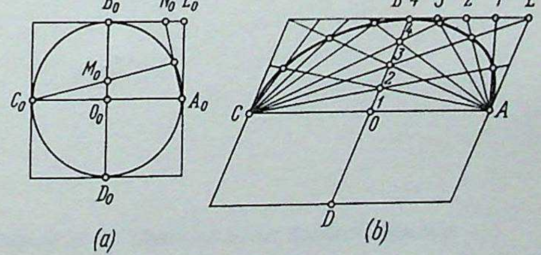

FIG. 67

The obtained axes of symmetry AB and CD are the axes of the ellipse.

5. Given a pair of conjugate diameters, construct an ellipse. We have already mentioned (Sec. 10) that 'to construct an ellipse' means to construct any number of points of the ellipse.

We are given two conjugate diameters AC and BD of an ellipse (Fig. 67*b*). Circumscribe a square about the periphery of the ellipse (Fig. 67*a*). Designate the points of tangency as A_0, B_0, C_0, and D_0. Suppose that A_0C_0 and B_0D_0 are the diameters that become conjugate diameters of the ellipse AC and BD after compression of the periphery. (Be careful! A_0C_0 and B_0D_0 are not the axes of compression.) Divide O_0B_0 into several congruent sections (Fig. 67 shows four such sections) and B_0E_0 into the same number of congruent sections. Count an equal number of sections from O_0 upward and from E_0 leftward. Thus, we obtain the points M_0 and N_0 respectively. Draw the lines C_0M_0 and A_0N_0, *the point of their intersection is contained in the periphery*. In fact, since the triangles $O_0M_0C_0$ and $E_0N_0A_0$ are congruent, $A_0N_0 \perp C_0M_0$.

After uniform compression the square transforms into a parallelogram. Therefore, while the segments OB and BE remain divided into congruent sections, OB and BE themselves are no longer congruent. The construction should be made as follows (Fig. 67*b*). Construct a parallelogram on the lines AC and BD as if they were centre lines. The segment OB is divided into congruent sections, and the points of division are designated 1, 2, 3, and so on. The segment EB is also divided into congruent sections, and the points of division are designated 1′, 2′, 3′, and so on. The points of intersection $C1 \cap A1'$, $C2 \cap A2'$, $C3 \cap A3'$, and so on are contained in the ellipse.

By studying the drawing closely, the reader should be able to construct the points of the ellipse in its other quarters.

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